16. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest to or farthest from the point (4, 2, 1).

17. A triangle is to be inscribed in the ellipse $\frac{1}{4}x^2 + y^2 = 1$ with one vertex of the triangle at $(-2, 0)$ and the opposite side perpendicular to the x axis. Find the largest possible area of the triangle.

18. Let x and y denote the acute angles of a right triangle. Find the maximum value of $\sin x \sin y$.

19. Let x, y, and z denote the angles of an arbitrary triangle. Find the maximum value of $\sin x \sin y \sin z$.

20. Find the minimum volume of a tetrahedron in the first octant bounded by the planes $x = 0, y = 0, z = 0$, and a plane tangent to the sphere $x^2 + y^2 + z^2 = 1$. (*Hint:* If the plane is tangent to the sphere at the point (x_0, y_0, z_0), then the volume of the tetrahedron is $1/(6x_0y_0z_0)$.)

21. A rectangular parallelepiped lies in the first octant, with three sides on the coordinate planes and one vertex on the plane $2x + y + 4z = 12$. Find the maximum possible volume of the parallelepiped.

Use Lagrange multipliers to solve Exercises 22–25, which also appear as maximum-minimum problems in Chapter 4.

22. A cylindrical can with bottom and no top has volume V. Find the radius of the can with the smallest possible surface area. (This is Exercise 33 in Section 4.6.)

23. Find the points on the parabola $y = x^2 + 2x$ that are closest to the point $(-1, 0)$. (This is Exercise 25 in Section 4.6.)

24. A rectangular printed page is to have margins 2 inches wide at the top and the bottom and margins 1 inch wide on each of the two sides. If the page is to have 35 square inches of printing, determine the minimum possible area of the page itself. (This is Exercise 31 in Section 4.6.)

25. An isosceles triangle is inscribed in a circle of radius r. Find the maximum possible area of the triangle. (This is Exercise 60 in Section 4.1.)

Lagrange multipliers can be used to find the extreme values of functions subject to more than one constraint. Suppose we wish to find the extreme values of a function f of three variables satisfying the two constraints

$$g_1(x, y, z) = c_1 \quad \text{and} \quad g_2(x, y, z) = c_2 \tag{21}$$

where c_1 and c_2 are constants. The method is to solve the equation

$$\text{grad} f(x, y, z) = \lambda \text{ grad } g_1(x, y, z) + \mu \text{ grad } g_2(x, y, z)$$

along with the constraints in (21) for (x, y, z) and for λ and μ if necessary and then to determine the largest and smallest values of f. Both λ and μ are called Lagrange multipliers. Use this method to solve Exercises 26–27.

26. Find the minimum distance between the origin and a point on the intersection of the paraboloid $z = \frac{3}{2} - x^2 - y^2$ and the plane $x + 2y = 1$.

27. Find the distance from the point $(2, -2, 3)$ to the intersection of the planes

$$2x - y + 3z = 1 \quad \text{and} \quad -x + 3y + z = -3$$

Applications

28. A construction company needs a type of funnel to reduce spillage when trucks are loaded with sand. The funnel is to consist of a circular cylinder with radius 3 feet on top of a cone with the same radius (Figure 13.57).
    a. If the entire funnel is to have a capacity (volume) of 300 cubic feet, find the heights H and h of the cylinder and cone that minimize the amount of material (surface area) needed.
    b. How, if at all, would the answer to part (a) be altered if the capacity were to be 400 cubic feet?

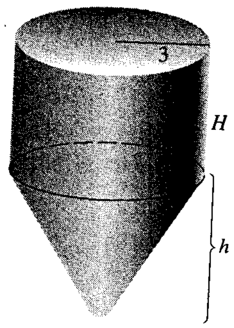

FIGURE 13.57 Figure for Exercise 28.

29. A rectangular storage box with volume 12 cubic inches is to be made in the form shown in Figure 13.58. Find the dimensions that will minimize the total surface area of the box. (*Hint:* Solve for λ in each equation you obtain.)

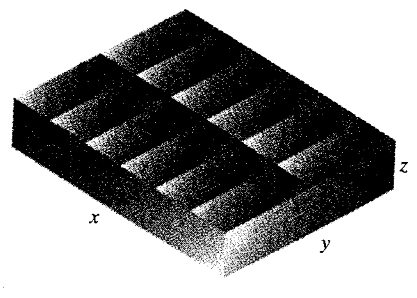

FIGURE 13.58 Figure for Exercise 29.

30. A rectangular box, open at the top, is to have a volume of 1728 cubic inches. Find the dimensions that will minimize the cost of the box if

a. the material for the bottom costs 16 times as much per unit area as the material for the sides.

b. the material for the bottom costs twice as much per unit area as the material for the sides.

31. In this exercise you will solve a realistic version of Exercise 36 in Section 4.6. A cylindrical pipe of radius r and length l must be slid on the floor from one corridor to a perpendicular corridor, each 3 meters wide (Figure 13.59). Find the dimensions of the pipe that maximize its volume V. (*Hint:* As in Figure 13.59, assume that the opposite ends of the pipe touch the walls when the angle between the pipe and the wall is $\pi/4$.)

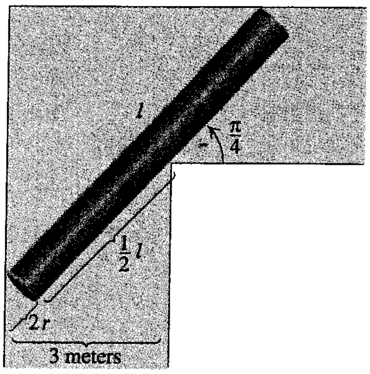

FIGURE 13.59 Figure for Exercise 31.

32. Suppose that on your vacation you plan to spend x days in San Francisco, y days in your home town, and z days in New York. You calculate that your total enjoyment $f(x, y, z)$ will be given by

$$f(x, y, z) = 2x + y + 2z$$

If plans and financial limitations dictate that

$$x^2 + y^2 + z^2 = 225$$

how long should each stay be to maximize your enjoyment?

33. The ground state energy $E(x, y, z)$ of a particle of mass m in a rectangular box with dimensions x, y, and z is given by

$$E(x, y, z) = \frac{h^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$$

where h is a constant. Assuming that the volume V of the box is fixed, find the values of x, y, and z that minimize the value of E.

34. The object distance p, image distance q, and focal length f of a simple lens satisfy the equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

Determine the minimum distance $p + q$ between the object and the image for a given focal length.

35. Fermat's Principle states that light always travels the path between points requiring the least time. Suppose light travels from a point A in one medium in which it has velocity v to a point B in a second medium in which it has velocity u. Using the fact that in a single medium, light travels in a straight line, and using Figure 13.60, show that the light is bent according to Snell's Law:

$$\frac{\sin\theta}{v} = \frac{\sin\phi}{u}$$

(*Hint:* The constraint is $x + y = l$, where x and y are the distances indicated in Figure 13.60.)

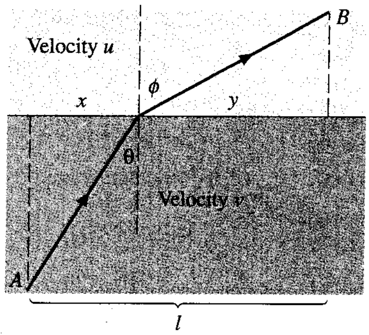

FIGURE 13.60 Figure for Exercise 35.

36. A pharmaceutical company plans to make capsules containing a given volume V of medicine. One executive would like to have the capsules in the form of a right circular cylinder having length h and base radius r with a hemisphere at each end (see Figure 13.61). A second executive objects to the wastefulness of materials, and contends that the same volume could be contained in a spherical capsule having a smaller surface area. Which executive should get the next promotion?

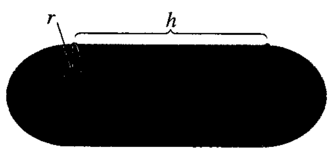

FIGURE 13.61 Figure for Exercise 36.

37. Let x represent capital and y labor in the manufacture of $f(x, y)$ units of a given product. Assume that capital costs a