Subspaces of the zero set of a polynomial

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Plan of talk

- 1. "Every party has a pooper, that's why we invited you..."
- 2. Background and statement of problem
- 3. A positive result
- 4. Negative results

2) Background and statement of problem

Definition

Let X be a real or complex Banach space and let $n \in \mathbb{N}$.
$P : X \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is an $n-$*homogeneous* (continuous) polynomial if $\exists A : X \times \cdots \times X \to \mathbb{K}$ with A being continuous and $n-$linear such that $P(x) = A(x, ..., x), \forall x \in X$.

Without loss, we can assume that the associated A is also *symmetric.* **Clearly**, each A gives rise to a unique P. Conversely, by the *polarization formula,* given an $n-$homogeneous polynomial P, one can recover the associated (unique) symmetric $n-$linear form A.

Example

1. $X = \mathbb{R}^n$, A an $n \times n$ real symmetric matrix, then $P : \mathbb{R}^n \to \mathbb{R},\ P(x) = xAx^T$ is a $2-$homogeneous polynomial.
2. $X = \ell_2$ and $P(x) = \sum_{j=1}^{\infty} a_j x_j^k$, where $(a_j) \in \ell_\infty$ and $k \geq 2$.

Problem

Let $P : X \to \mathbb{K}$ *be an* $n-$*homogeneous polynomial,* $P(0) = 0$.
What can we say about the lineability (= spaceability) of $P^{-1}(0)$?

Example

1. $X = \mathbb{R}^n, P(x) = \sum_j x_j^2 \Rightarrow P^{-1}(0) = \{0\}$.
2. $X = \mathbb{C}^n, P(x) = \sum_j x_j^2 \Rightarrow P^{-1}(0) \supset span\{e_1 + ie_2, e_3 + ie_4, ...\}$.

So, roughly, $P^{-1}(0)$ contains a vector space of dimension $n/2$.

3. $X = \mathbb{R}^n, P(x) = \sum_j x_j^3 \Rightarrow P^{-1}(0)$ again contains a "big" subspace.

Conclude: Lineability of $P^{-1}(0)$ depends on field $\mathbb{K}$ and on homogeneity of P.

3) A positive result

Theorem

(A. Plichko & A. Zagorodnyuk) Let X be an infinite dimensional complex Banach space. Let $P : X \to \mathbb{C}$ be an $n-$homogeneous polynomial. Then $P^{-1}(0)$ contains an infinite dimensional subspace.

We'll sketch the proof, following an argument of RMA & P. Rueda of a restated version of the above theorem.

Theorem

Let $m, d \in \mathbb{N}$ be given. Then $\exists n = n(m, d)$, such that if $P : \mathbb{C}^n \to \mathbb{C}$ is a $d-$homogeneous polynomial, then $P^{-1}(0)$ contains an $m-$dimensional subspace.

PROOF Idea: Induction. If $d = 1$, then we're talking about *linear* mappings $\mathbb{C}^n \to \mathbb{C}$. So, $n = m + 1$ works.

If $d = 2$ and m are given, let's determine the number n of variables so that every 2−homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is $\equiv 0$ on an m−dimensional subspace of $\mathbb{C}^n$.

Easy fact: If $n \geq 2$, then $\exists x_1 \in \mathbb{C}^n,\ x_1 \neq 0$, such that $P(x_1) = 0$. Call $S(x_1) = \{x \in \mathbb{C}^n \mid A(x_1, x) = 0\}$.

$S(x_1) \cong [x_1] \bigoplus \mathbb{C}^{n-2}$. If $n - 2 \geq 2$, i.e. if $n \geq 4$, then the easy fact $\Rightarrow \exists x_2 \in \mathbb{C}^{n-2}, x_2 \neq 0$, with $P(x_2) = 0$.

Then $[x_1, x_2] \subset P^{-1}(0)$. Indeed $P(a_1x_1 + a_2x_2) = A(a_1x_1 + a_2x_2, a_1x_1 + a_2x_2) = a_1^2P(x_1) + 2a_1a_2A(x_1, x_2) + a_2^2P(x_2) = 0$. Call $S(x_2) = \{x \in \mathbb{C}^{n-2} \mid A(x_2, x) = 0\}$. So, $S(x_2) \cong [x_2] \bigoplus \mathbb{C}^{n-4}$, and if $n - 4 \geq 2$, i.e. $n \geq 6$, then $\exists x_3 \in \mathbb{C}^{n-4}$ with $P(x_3) = 0$. Then $[x_1, x_2, x_3] \subset P^{-1}(0)$, etc.

To start the case of $d = 3$, i.e. $3-$homogeneous polynomials P on $\mathbb{C}^n$, if $n \geq 2$, then $\exists x_1 \neq 0$ such that $P(x_1) = 0$.
Use the $d = 1$ and $d = 2$ cases to find $x_2 \in \mathbb{C}^n$ $(n \geq 6)$ so that $P(x_2) = A(x_1, x_1, x_2) = A(x_1, x_2, x_2) = 0$. ... etc. ... $\square$

Remark: In particular, this method works in *infinite dimensions. Namely:*

Corollary

Let X be a complex Banach space, dim $X = \infty$. Let $P : X \to \mathbb{C}$ be a polynomial, $P(0) = 0$. Then $\exists Y \subset X$, dim $Y = \infty$, such that $P|_Y \equiv 0$.

We'll return to this later.

Question: What about a positive result in the case of *real* spaces?
First answer: Silly question! Take $P : \mathbb{R}^2 \to \mathbb{R}$,
$P(x_1, x_2) = x_1^2 + x_2^2$, or more generally $P : \ell_2 \to \mathbb{R}$, $P(x) = \sum_j x_j^2$.
Second answers: (i) What about **odd**-homogeneous polynomials, e.g. $P(x_1, x_2) = x_1^3 - x_2^3$?
(ii) What about $2-$homogeneous polynomials on **very big** real Banach spaces? In connection with (ii),

Definition

A polynomial $P : X \to \mathbb{R}$ (X a real Banach space) is called *positive definite* if $P(x) \geq 0$ $(\forall x \in X)$ and $P(x) = 0 \iff x = 0$.

Remark X admits a positive definite $2-$homogeneous polynomial $\iff$ X admits a continuous injection $X \to \ell_2$.

Theorem

(RMA, C. Boyd, R. Ryan, & I. Zalduendo) Assume that the (automatically non-separable) real Banach space X does not admit any positive definite 2-homogeneous polynomial. Then for every $2-$homogeneous polynomial $P : X \to \mathbb{R}$, $P^{-1}(0)$ contains an infinite dimensional subspace.

Corollary

Assume that X does not admit any positive definite $4-$homogeneous polynomial. Then every $3-$homogeneous polynomial $P : X \to \mathbb{R}$ is such that $P^{-1}(0)$ contains a non-separable subspace of X.

4) Negative results

Return to complex case. Let $P : X \to \mathbb{C}$ be an $n-$homogeneous polynomial, $X =$ *infinite dimensional* complex Banach space. We know: $\exists Y \subset X$, dim $Y = \infty$, such that $P|_Y \equiv 0$.

Problem

What if X is a **non-separable** *complex Banach space. Can we then assert that $\exists Y$,* **non-separable,** *with $P|_Y \equiv 0$?*

No!

Theorem

(A. Avilés & S. Todorcevic) $\exists$ a $2-$homogeneous polynomial $P : \ell_1(\Gamma) \to \mathbb{C}$ (Γ a 'big' index set) with the property that if $Y \subset \ell_1(\Gamma)$ is such that $P|_Y \equiv 0$, then Y is separable.

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(More about 'size' of zero-subspace $\subset P^{-1}(0)$)

Definition

Let $P: X \to \mathbb{K}$ be an $n-$homogeneous polynomial. A subspace $Y \subset P^{-1}(0)$ is *maximal* means $\nexists\ Z,\ Z \underset{\neq}{\supseteq} Y$, with $P|_Z \equiv 0$.

Theorem

(A. Avilés & S. Todorcevic) $\exists$ *a polynomial* $P : \ell_1(\Gamma) \to \mathbb{C}$ *(Γ a 'big' index set) such that* $P^{-1}(0)$ *has both maximal separable and maximal non-separable subspaces.*

Proof (idea): First, **Exercise**: $\exists$ a collection $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ satisfying

- $\forall A \in \mathcal{C}, |A| = \infty,$
- $\mathcal{C}$ is uncountable, and
- $\forall A_1 \neq A_2$ in $\mathcal{C},\ A_1 \cap A_2$ is finite.

Let $\Gamma = \mathcal{C} \cup \mathbb{N}$ (*disjoint* union!). Define $P : \ell_1(\Gamma) \to \mathbb{C}$ by

$$P(x, y) = \sum_{n \in \mathbb{N}, A \in \mathcal{C}} x_n y_A.$$

(Here $(x, y) = ((x_n)_n, (y_A)_A) \in \ell_1(\Gamma)$.)
One shows $\{(x, 0) \mid x = (x_n) \in \ell_1\}$ is a maximal (separable) space in $P^{-1}(0)$. Since $P \equiv 0$ on $\{(0, y) \mid y \in \ell_1(\mathcal{C})\}$ (non-separable), proof is complete. $\square$.

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Odd-homogeneous polynomials on *real* Banach spaces.

Theorem

(RMA & P. Hájek) Let $k, n \in \mathbb{N}$, *with* n *odd. Then* $\exists N = N(k, n)$ *such that* $\forall P : \mathbb{R}^N \to \mathbb{R}$, P *being* $n-$*homogeneous,* $\exists Y \subset \mathbb{R}^N, dim Y = k$, *such that* $P|_Y \equiv 0$.

Remark: Not very good estimates on $N = N(k, n)$. However,

Theorem

(RMA & P. Hájek) For every real Banach space X, *dim* $X = \infty$, *for every odd* $n \in \mathbb{N}$, *there is an* $n-$*homogeneous polynomial* $P : X \to \mathbb{R}$, *such that* $P^{-1}(0)$ *contains no infinite dimensional vector space.*

In fact, our P satisfies: $\forall x \in X, x \neq 0$, $\exists k \in \mathbb{N}$ such that if $Y \subset X$ is such that $x \in Y \subset P^{-1}(0)$, then dim $Y \leq k$.