Small and big sets in analysis

Frédéric Bayart

Université Blaise Pascal, Clermont-Ferrand

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What is a small set?

Aaronszajn null, capacity zero, cube null, Dirichlet, first category, Γ-null, Gauss null, Haar null, Helson, HP-null, measure zero, removable sets for bounded analytic functions, sets with small Hausdorff dimension, σ-porous, U-sets,...

Baire proves his famous theorem in a course that he gave in "Collège de France" in 1903/1904 and which was published in 1904.
Let us quote Baire.

Un sous-ensemble M de la droite est non dense dans un intervalle PQ si, étant donné un sous-intervalle ouvert arbitraire AB de PQ, le complémentaire de M dans AB contient un sous-intervalle ouvert.

Nous dirons qu'un ensemble est de première catégorie dans un intervalle PQ s'il est contitué par la réunion d'une infinité dénombrable d'ensembles dont chacun est non-dense dans PQ.

Je dis que si G est un ensemble de première catégorie sur un segment PQ, il y a dans toute portion de PQ des points qui n'appartiennent pas à G.

In other words, a first category subset of $\mathbb{R}$ has empty interior!

Car G est formé d'une infinité dénombrable d'ensembles non-denses $G_1, G_2, \ldots$. Soit ab un intervalle pris sur PQ. L'ensemble G_1 étant non-dense dans PQ, il est possible de déterminer dans ab une portion a_1b_1 ne contenant aucun point de G_1. De même, dans a_1b_1, il est possible de déterminer une portion a_2b_2 ne contenant aucun point de G_2, et ainsi de suite. Nous formons ainsi une suite d'intervalles $a_1b_1, a_2b_2, \ldots$ dont chacun est contenu dans le précédent et tels que a_nb_n ne contient aucun point de $G_1, G_2, \ldots$. Il existe un point A appartenant à tous ces intervalles. Ce point n'appartient pas à G puisqu'il ne peut appartenir à aucun des ensembles $G_1, G_2, \ldots$. La proposition est donc démontrée.

Applications of Baire theorem - regularity

Theorem (Baire, 1904)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The following are equivalent:

1. *For any $\varnothing \neq F \subset \mathbb{R}^n$ closed, $f_{|F}$ admits at least one point of continuity.*
2. *There exists a sequence (f_l) of continuous functions such that*

$$\forall x \in \mathbb{R}^n,\ f(x) = \lim_{l \to +\infty} f_l(x).$$

Applications of Baire theorem - regularity

- A separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ has points of continuity.
- A convex function $f : X \to \mathbb{R}$ with X a separable Banach space with X^* separable has points of Fréchet-differentiability.

Let $f : \mathcal{U} \subset X \to Y$ and $x_0 \in \mathcal{U}$. We say that f is Gateaux-differentiable at x_0 if there exists $T \in \mathcal{L}(X, Y)$ such that for every $u \in X$,

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = Tu.$$

If the limit exists uniformly in u on the unit sphere of X, we say that f is Fréchet-differentiable at x_0.

Applications of Baire theorem - Uniformity

- Uniform boundedness principle: Let X, Y be Banach spaces and let $(T_i)_{i\in I}$ be a family of $\mathcal{L}(X,Y)$ such that, for any $x \in X$, $\sup_{i\in I} \|T_i x\| < +\infty$. Then $\sup_{i\in I} \|T_i\| < +\infty$.
- Let $f \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ so that, for any $x \in \mathbb{R}$, there exists $n(x)$ with $f^{n(x)}(x) = 0$. Then f is a polynomial.

Applications of Baire theorem - Condensation of singularities

Contrapositive of the uniform boundedness principle.

- There exists a residual set of function $f \in \mathcal{C}(\mathbb{T})$ such that the Fourier series of f diverges at any point of a residual subset of $\mathbb{T}$.
- Any function in a residual subset of $\mathcal{C}([0,1])$ is nowhere differentiable.
- Existence of Besicovitch sets: there exist (plenty of) closed subsets of $\mathbb{R}^2$ with measure 0 containing a unit line segment in each direction. (Besicovitch, 1928, Körner, 2003).

Applications of Baire theorem - non void implies big

An operator T on a Banach space X is called hypercyclic provided there exists a vector $x \in X$ such that its orbit $\{T^n x : \ n \geq 0\}$ is dense in X.
We denote by $HC(T)$ the set of hypercyclic vectors for T.
As soon as $HC(T)$ is nonempty, it is a residual subset of X.

Smallness, a relative notion

The real line can be decomposed into a set of measure 0 and a set of first category.

$\mathbb{Q} = (q_k)_{k\geq 1}$, $I_{k,l} = \left(q_k - \frac{1}{2^{k+l}}, q_k + \frac{1}{2^{k+l}}\right)$,
$G_l = \bigcup_{k=1}^{+\infty} I_{k,l}$, $B = \bigcap_{l\geq 1} G_l$.

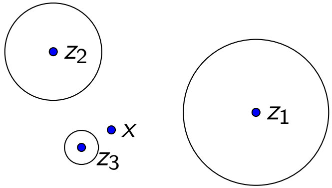
Definition - porous sets

Let (X, d) be a metric space. A set $E \subset X$ is nowhere dense if and only if $\forall x \in E,\ \forall \varepsilon > 0,\ \exists z \in X \backslash E,\ \exists \delta > 0,$

$$d(x, z) < \varepsilon \text{ and } B(z, \delta) \cap E = \varnothing.$$

Let $\lambda \in (0, 1)$, $E \subset X$ and $x \in E$. We say that E is **λ-porous** at x if $\forall \varepsilon > 0,\ \exists z \in X \backslash E,$

$$d(x, z) < \varepsilon \text{ and } B\big(z, \lambda d(x, z)\big) \cap E = \varnothing.$$



Properties of porous sets

A set $E \subset X$ is porous if it is porous at each of its points. It is λ-porous if it is λ-porous at each of its points, namely if $\forall x \in E,\ \forall \varepsilon > 0,\ \exists z \in X \setminus E,$

$$d(x,z) < \varepsilon \text{ and } B\big(z, \lambda d(x,z)\big) \cap E = \varnothing.$$

Proposition

A porous set $E \subset \mathbb{R}^n$ is nowhere dense and has Lebesgue measure zero.

For almost every $x \in E$,

$$\lim_{\varepsilon \to 0} \frac{\mu\big(E \cap B(x,\varepsilon)\big)}{\mu\big(B(x,\varepsilon)\big)} = 1.$$

σ-porous sets

A set $E \subset X$ is σ-porous if it is a countable union of porous sets.

Corollary

A σ-porous set is meager and has Lebesgue measure zero.

Proposition

The notion of σ-porosity is a strict refinement of the notions of Lebesgue measure 0 sets and of sets of first category.

Write $\mathbb{R}^n = A \cup B$ with A meager and B with Lebesgue measure 0.

Proposition

There is a non-σ-porous set in $\mathbb{R}^n$ which has measure 0 and is meager.

Argument : Let $A \subset \mathbb{R}^n$ and assume that A is not of the first category or is not of measure 0. Then $A + A$ contains a non-empty open set. There exists a non-σ-porous set $A \subset \mathbb{R}^n$ such that, for every finite sequence $(c_1, \dots, c_n)$, the set $\sum_{j=1}^{n} c_j A$ is of measure 0. (Tkadlec, 1983).

Example - Cantor set

Let $\alpha := (\alpha_n)$ be a sequence with $0 < \alpha_n < 1$. Let $C(\alpha) \subset [0,1]$ be the associated symmetric Cantor set. At the n-th step, we delete from the 2^{n-1} remaining intervals of length d_n a concentric interval of size $\alpha_n d_n$.

Observation : $C(\alpha)$ has measure 0 if and only if $\sum_{n\geq 1} \alpha_n = +\infty$.

Theorem (Humke,Thomson (1985))

$C(\alpha)$ is non σ-porous if and only if $\alpha_n \to 0$.

In particular, for $\alpha_n = \frac{1}{n+1}$, we get an example of a measure 0 set which is not σ-porous.

An example from number theory

For $x \in (0,1)$ and $k \geq 1$, let $a_k(x)$ be its k-th digit in its decimal expansion.

Theorem (Foran, 1985)

The set

$$E = \left\{ x \in X;\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \frac{\#\{k \leq n;\ a_k(x) = 1\}}{n} \in [1/4, 3/4] \right\}$$

is a first category set which is not σ-porous.

Remark : any Banach space supports a first category set which is not σ-porous.

A small history

The first to introduce porous sets (with a different terminology) is Denjoy (1920-1941).

Theorem

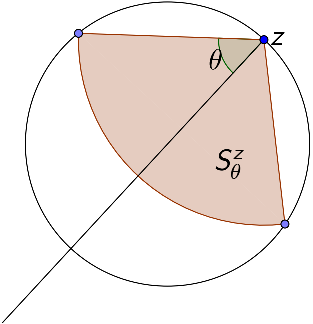
Let P be a perfect nowhere dense subset of $\mathbb{R}$ and let $\lambda \in (0,1)$. Then the sets of points in P at which P is not λ-porous is a first category subset of P.

Denjoy applied this result to the second symmetric derivative of a function $F : (a, b) \to \mathbb{R}$.

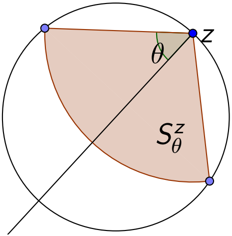
A small history

The first to introduce σ-porous sets was Dolženko (1967). He applied this notion to cluster sets of functions. Let $f:\mathbb{D}\to\mathbb{C}$, let $\theta\in(0,\pi/2)$, let $z\in\mathbb{T}$ and let S_θ^z be the corresponding Stolz angle. The Stolz cluster set associated to f and S_θ^z is defined as

$$C(f,\theta,z)=\{\ell\in\mathbb{C};\ \exists(z_n)\in S_\theta^z \text{ s.t.} z_n\to z \text{ and } f(z_n)\to\ell\}.$$



Dolženko theorem



We say that z is singular if there exist $\theta_1 \neq \theta_2$ such that $C(f, \theta_1, z) \neq C(f, \theta_2, z)$.

Theorem

- *The set of singular points is a σ-porous subset of $\mathbb{T}$.*
- *Given a σ-porous subset E of $\mathbb{T}$, there exists a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that every $z \in E$ is a singular point for f.*

Examples of σ-porous sets - a general method

How to prove that a set is σ-porous???

Proposition (Olevskii, 1991)

Let E be a convex nowhere dense set in a Banach space X. Then E is $1/2$-porous.

proof

Corollary

The set of function which have convergent Fourier series at a specified point is σ-porous in $\mathcal{C}([-\pi, \pi])$.

Other applications - Fréchet differentiability

Theorem (Preiss, Zajíček (1984))

Let X be a separable Banach space such that X^* is separable and let f be a continuous convex function defined on an open subset of X. Then the set of points of non-Fréchet differentiability of f is σ-porous.

Other applications

Well-posed optimization problems :

Theorem (Deville, Revalski (2000))

Let X be a "smooth" Banach space and let Y be the Banach space of Lipschitz and $\mathcal{C}^1$-smooth functions on X. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper bounded from below lower semi-continuous function. Then set

$$T = \{g \in Y;\ f + g \ \textit{attains its minimum}\}$$

has a complement which is σ-porous in Y.

In the previous theorem, "smooth" means that there exists a Lipschitz and $\mathcal{C}^1$-function $b : X \to \mathbb{R}$ which is not identically equal to zero and which has bounded support.

A Counter-example

Theorem (B. (2005))

Let B be the backward shift on $\ell^p(\mathbb{N})$ or on $c_0(\mathbb{N})$. Then $[HC(2B)]^c$ is not σ-porous.

Theorem (Foran's lemma (1984))

Let $\mathcal{F}$ be a nonempty family of nonempty closed sets. Assume that for each $F \in \mathcal{F}$ and each open ball $B(x,r)$ with $F \cap B(x,r) \neq \varnothing$, there exists $G \in \mathcal{F}$ such that

- $G \cap B(x,r) \neq \varnothing$
- $G \cap B(x,r) \subset F \cap B(x,r)$
- *$F \cap B(x,r)$ is $1/2-$porous at no point of $G \cap B(x,r)$.*

Then no set from $\mathcal{F}$ is $\sigma-$porous.

An infinite-dimensional version of measure 0 set

On $\mathbb{R}^n$, the Lebesgue measure plays a particular role: it is invariant by translation.

On an infinite dimensional space, there does not exist a non-zero measure which is finite on balls and which is invariant by translation. Which properties of sets of Lebesgue measure 0 are important?

- If A has measure 0, any translate and any dilate of A has measure 0.
- A countable union of negligible sets is negligible.
- The complement of a negligible set is dense.

A further property: if X is infinite-dimensional, a compact set should be negligible.

Haar null sets

Let X be an abelian group endowed with a translation invariant metric d with respect to which X is complete and separable.

Definition (Christensen (1972))

A Borel set $A \subset X$ is called a Haar null set if there is a probability measure μ on X such that $\mu(x + A) = 0$ for any $x \in X$. μ is then called a transverse measure for A.
A subset of X is Haar null if it is contained in a Borel Haar null set.
The complement of a Haar null set is called prevalent.

Haar null sets are also called sometimes shy sets (Hunt, Sauer, Yorke, 1992).

Why this terminology?

Proposition

Assume that X is locally compact. Then $A \subset X$ is Haar null if and only if its Haar measure is equal to 0.

$\Leftarrow$: let μ_0 be the Haar measure on X. If $\mu_0(A) = 0$, then $\mu_0(A + x) = 0$ for any $x \in X$ and $\mu(A + x) = 0$ for any measure μ which is absolutely continuous with respect to μ_0.

$\Rightarrow$: Assume that $\mu(A + x) = 0$ for any $x \in X$ with μ a probability measure. By Fubini's theorem

$$\begin{aligned}\mu_0(A) &= \int \mu_0(A + x) d\mu(x) = \int \int \chi_A(x + y) d\mu(x) d\mu_0(y) \\ &= \int \mu(A + y) d\mu_0(y) = 0.\end{aligned}$$

The required properties

- A translate of a Haar null set is Haar null.
- If X is a vector space, a dilate of a Haar null set is Haar null.
- A prevalent set is dense.
- A countable union of Haar null sets is Haar null.
- If X is not locally compact, any compact subset of X is Haar null.

The union of two Haar null sets is Haar null

Lemma

Let μ and ν be measures. If μ is transverse to a Borel set A, so is $\mu \star \nu$.

Thus, if μ is a probability measure which is transverse to A and ν is a probability measure which is transverse to B, then $\mu \star \nu$ is a probability measure which is transverse to $A \cup B$.

$$\mu \star \nu(S) = \int \mu(S - y) d\nu(y) = \int \nu(S - z) d\mu(z).$$

$$\mu \star \nu(A + x) = \int \mu(A - y + x) d\nu(y) = 0.$$

How to prove that a set is Haar null (or prevalent)?

Theorem (Hunt, 1994)

The set of nowhere Lipschitz function form a prevalent subset of $\mathcal{C}([0,1])$.

Method 1 : use a probe space.

Definition

Let X be a Banach space. We call a finite-dimensional subspace $P \subset X$ a probe space for a set $M \subset X$ if Lebesgue measure on P is transverse to a Borel set containing the complement of M.

A set M admitting a probe P is prevalent. We say sometimes that is k-prevalent, with $k = \dim(P)$.

Probe space I

The set of nowhere Lipschitz function form a prevalent subset of $\mathcal{C}([0,1])$.

Can we use a one-dimensional probe space?

This would imply that there exists a function $g \in \mathcal{C}([0,1])$ such that, for any $f \in \mathcal{C}([0,1])$, $f + \lambda g$ is nowhere Lipschitz for almost every $\lambda \in \mathbb{R}$. Pick any g and define $f = -xg$. Then $f + \lambda g = (\lambda - x)g$ is always differentiable at $x = \lambda$.

A two-dimensional probe space

The set of nowhere Lipschitz function form a prevalent subset of $\mathcal{C}([0,1])$.

Let $g(x) = \sum_{k\geq 1} \frac{1}{k^2}\cos(2^k\pi x)$, $\quad h(x) = \sum_{k=1}^{+\infty} \frac{1}{k^2}\sin(2^k\pi x)$.

Lemma

There exists $c>0$ such that, for any $\alpha,\beta\in\mathbb{R}$ and any closed interval $I\subset[0,1]$ with length $\varepsilon>0$, then

$$\max_I(\alpha g+\beta h)-\min_I(\alpha g+\beta h)\geq c\frac{\sqrt{\alpha^2+\beta^2}}{(\log\varepsilon)^2}.$$

For $M>0$ and $x\in[0,1]$, we say that f is M-lipschitz at x if for any $y\in[0,1]$,

$$|f(x)-f(y)|\leq M|x-y|.$$

A two-dimensional probe space

Let $M \geq 1$ and $f \in \mathcal{C}([0,1])$. We define

$$S_M = \{(\alpha, \beta) \in \mathbb{R}^2;\ f + \alpha g + \beta h \text{ is } M-\text{lipschitz at some } x \in [0,1]\}.$$

Let $N \geq 1$ and cover $[0,1]$ by N closed intervals of length $\varepsilon = 1/N$. Let I be one of this intervals and let

$$S_{M,I} = \{(\alpha, \beta) \in \mathbb{R}^2;\ f + \alpha g + \beta h \text{ is } M-\text{lipschitz at some } x \in I\}.$$

Let $(\alpha_1, \beta_1),\ (\alpha_2, \beta_2) \in S_{M,I}$. Then $f_1 = f + \alpha_1 g + \beta_1 h$ (resp. $f_2 = f + \alpha_2 g + \beta_2 h$) is M-lipschitz at some x_1 (resp. x_2). Thus,

$$|f_i(x) - f_i(x_i)| \leq M|x - x_i| \leq M\varepsilon.$$

By the triangle inequality,

$$|(f_1(x) - f_2(x)) - (f_1(x_1) - f_2(x_2))| \leq 2M\varepsilon.$$

A two-dimensional probe space

We have

$$|(f_1(x) - f_2(x)) - (f_1(x_1) - f_2(x_2))| \leq 2M\varepsilon$$

hence

$$\max_I(f_1 - f_2) - \min_I(f_1 - f - 2) \leq 4M\varepsilon.$$

Now, $f_1 - f_2 = (\alpha_1 - \alpha_2)g + (\beta_1 - \beta_2)h$. By the lemma,

$$\sqrt{(\alpha_1^2 - \alpha_2^2) + (\beta_1^2 - \beta_2)^2} \leq C'\varepsilon(\log \varepsilon)^2.$$

Hence, $S_{M,I}$ is contained in a disk of radius $\varepsilon(\log \varepsilon)^2$ and

$$\mathrm{measure}(S_M) \leq C''N\varepsilon^2(\log \varepsilon)^4 = C''\varepsilon(\log \varepsilon)^4.$$

Since ε can be arbitrarily small, $\mathrm{measure}(S_M) = 0$.

The stochastic process method

To prove that A is prevalent, it suffices to exhibit a random variable $U : (\Omega, \mathcal{F}, \mathbb{P}) \to X$ such that

$$\forall f \in X, \text{ a.s. } f + U \in A.$$

Here, we can choose for U the Brownian motion B, since

$$\forall f \in \mathcal{C}([0,1]), \ \forall \varepsilon > 0, \ f + B \text{ is nowhere } \mathcal{C}^{1/2+\varepsilon}.$$

Beyond

Theorem (Hunt (1994))

A prevalent function in $\mathcal{C}([0,1])$ is nowhere $\mathcal{C}^\gamma$ for any $\gamma > 0$.

Proof:

- We show that for a fixed $\gamma > 0$, a prevalent function in $\mathcal{C}([0,1])$ is nowhere $\mathcal{C}^\gamma$ (two methods...)
- Countable intersection!

The lack of Fubini's theorem

Be careful! Fubini's theorem becomes false for Haar null sets!

$A = \left\{(f,t) \in L^2(\mathbb{T}) \times \mathbb{T};\ \text{the Fourier series of } f \text{ converges at } t\right\}.$

$$\begin{aligned} A_{f_0} &= \{t \in \mathbb{T};\ (f_0,t) \in A\} \\ A^{t_0} &= \{f \in L^2(\mathbb{T});\ (f,t_0) \in A\}. \end{aligned}$$

A_{f_0} is prevalent (for any $f_0 \in L^2(\mathbb{T})$).
A^{t_0} is Haar null (for any $t_0 \in \mathbb{T}$). Indeed, let g be any function whose Fourier series diverges at t_0. Then for any $h \in L^2(\mathbb{T})$, $\{\lambda;\ h + \lambda g \in A^{t_0}\}$ contains at most one point.
What about A? It is prevalent! (exercise!)

σ-porosity vs Haar null sets

In finite-dimensional spaces, σ-porous sets are Haar null.

Theorem (Preiss - Tišer (1995))

Each separate infinite-dimensional Banach space X can be decomposed into two Borel subsets $X = A \cup B$ such that the intersection of A with any line in E has (one-dimensional) measure zero and B is a countable union of closed σ-porous sets.

In particular, there exists a closed porous set in X which is not Haar null.

Some examples

- Gateaux-differentiability of a Lipschitz function (Christensen, 1973): let X be a separable Banach space. Let f be a real valued Lipschitz function defined on an open set $U \subset X$. Then the set of points in U where f is not Gateaux differentiable is Haar null.
- Dynamical systems (Hunt, Sauer, Yorke 1992): for any $k \geq 1$, a prevalent function $f \in \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^n)$ has the property that all of its periodic points are hyperbolic.
- Multifractal analysis (Fraysse, Jaffard, 2006): Let $f \in \mathcal{C}(\mathbb{R}^d)$. Let $x \in \mathbb{R}^d$ and let $h_f(x)$ be the Hölder exponent of f at x. For $h \geq 0$, let $d_h(f) = \dim_{\mathcal{H}}\{x;\ h_f(x) = h\}$. If $s - d/p \geq 0$, then a prevalent function $f \in B_p^{s,q}(\mathbb{R}^d)$ satisfies $d_f(h) = hp - sp + d$ for any $h \in [s - d/p, s]$.

HP-small sets

Let X be a (separable) Banach space, $E \subset X$, $\lambda \in (0,1)$.

E is λ-porous if

$$\forall x \in E, \forall \varepsilon > 0, \exists z \in X \setminus E, \|x-z\| \leq \varepsilon \text{ and } B\big(z, \lambda\|x-z\|\big) \cap E = \varnothing.$$

E is λ-lower porous if

$$\forall x \in E, \ \forall \varepsilon > 0, \ \exists z \in X \setminus E, \|x - z\| \leq \varepsilon \text{ and } B\big(z, \lambda\varepsilon\big) \cap E = \varnothing.$$

Definition (Kolar, 2001)

E is HP_λ if for any $\varepsilon > 0$ there exist $K > 0$ and a sequence of balls $\big(B_n = B(y_n, \lambda\varepsilon)\big)$ such that $\|y_n\| \leq \varepsilon$ and, for any $x \in X$,

$$\mathrm{card}\{n \in \mathbb{N};\ (x + B_n) \cap E \neq \varnothing\} \leq K.$$

E is **HP-small** if there is some $\lambda \in (0,1)$ such that E is a countable union of sets with property HP_λ.

HP-small sets

Theorem (Kolar, 2001)

A HP-small set is Haar null.

Lemma (Matoušková, 1998)

A set $E \subset X$ is Haar null if and only if for every $\delta > 0$ and every $\varepsilon > 0$, there exists a Borel probability measure μ with support contained in $\bar{B}(0, \varepsilon)$ and $\mu(E + x) \leq \delta$ for every $x \in X$.

Let E be a closed HP_λ-set. Let $\varepsilon > 0$ and $K > 0$, $(B_n = B(y_n, \lambda\varepsilon))$ a sequence of balls as in the definition:

$$\forall x \in X,\ \mathrm{card}\{n \in \mathbb{N};\ (x + B_n) \cap E \neq \varnothing\} \leq K.$$

Let $n > K/\delta$ and define $\mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i}$.

HP-small sets

Theorem (Kolar, 2001)

The set of nowhere Lipschitz functions is HP-small in $\mathcal{C}([0,1])$.

Γ-null sets

Let $T = [0,1]^{\mathbb{N}}$ endowed with the product topology and product Lebesgue measure μ. Let X be a separable Banach space and $\Gamma(X)$ be the space of continuous mappings $T \to X$ having continuous partial derivatives.

Definition (Lindenstrauss, Preiss, 2003)

A Borel set $E \subset X$ is called Γ-null if

$$\{\gamma \in \Gamma(X);\ \mu(\gamma^{-1}(E)) > 0\}$$

is a first category subset of $\Gamma(X)$.

Translation invariant, coincide with Lebesgue measure 0 on $\mathbb{R}^n$, generally not comparable with Haar null sets.

Γ-null sets

Let X be a separable Banach space. Let f be a real valued Lipschitz function defined on an open set $U \subset X$. Then the set of points in U where f is not Gâteaux differentiable is Haar null.

Theorem (Lindenstrauss, Preiss 2003)

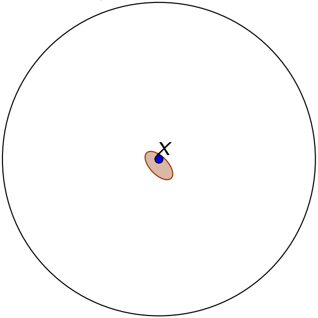
Every real-valued Lipschitz function on a Banach space X with separable dual is Fréchet differentiable Γ-almost everywhere provided that every porous set in X is Γ-null.

This last assumption is true on c_0 and false on ℓ^2.

Further reading

- Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional Analysis*, AMS Colloquium Publications 48 (2000).
- L. Zajíček, *Porosity and $\sigma-$porosity*, Real Analysis Exchange 13 (1987).
- L. Zajíček, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. 5 (2005).
- B. Hunt, T. Sauer and J. Yorke, *Prevalence: a translation-invariant almost every on infinite dimensional spaces*, Bull. AMS 27 (1992).
- J. Lindenstrauss, D. Preiss, David and J.Tišer, *Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces*, Annals of Mathematics Studies, 179 (2012).

A convex nowhere dense set E in a Banach space X is 1/2-porous.



Let $x \in X,\ \varepsilon > 0$.

Step 1. There exists $\phi \in X^*$, $\|\phi\| = 1$, $c > 0$ and $z \in B(x, \varepsilon/4)$ such that $\phi_{|E} < c$ and $\phi(z) \geq c$. Let $x \in X,\ \varepsilon > 0$.