Dense-lineability in classes of ultradifferentiable functions

Céline ESSER
Celine.Esser@ulg.ac.be

University of Liège – Institute of Mathematics

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Introduction

Recall. A function $f \in C^\infty(\Omega)$ is analytic at $x_0 \in \Omega$ if there exist a compact neighborhood K of x_0 and two constants $C, h > 0$ such that

$$\sup_{x\in K} \left| D^k f(x) \right| \leq C h^k k! \quad \forall k \in \mathbb{N}_0.$$

Question

How large is the set of nowhere analytic functions in the Fréchet space $\mathcal{C}^\infty([0,1])$?

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2. Prevalence
3. Lineability and algebrability

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1. Residuality
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3. Lineability and algebrability

Lineability (Aron, Gurariy, Seoane-Sepúlveda 2005)

Let X be a topological vector space, M a subset of X, and μ a cardinal number.

(1) The set M is lineable if $M \cup \{0\}$ contains an infinite dimensional vector subspace. If the dimension of this subspace is μ, M is said to be μ-lineable.

(2) When the above linear space can be chosen to be dense in X, we say that M is (μ-)dense-lineable.

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Algebrability (Aron, Pérez-García, Seoane-Sepúlveda 2006)

Let $\mathcal{A}$ be an algebra and M be a subset of $\mathcal{A}$.

(1) The set M is algebrable if $M \cup \{0\}$ contains a subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that the cardinality of any system of generators of $\mathcal{C}$ is infinite. If the cardinality of this system is μ, M is said to be μ-algebrable.

(2) If $\mathcal{A}$ is endowed with a topology and if the subalgebra $\mathcal{C}$ can be taken dense in $\mathcal{A}$, we say that M is (μ-)dense-algebrable.

Results

The set of nowhere analytic functions is residual, prevalent and $\mathfrak{c}$-algebrable in the space $\mathcal{C}^\infty([0,1])$.

- Morgenstern 1954
- Cater 1984
- Salzmann and Zeller 1955
- Bernal-Gonzalez 2008
- Bastin et al. 2012
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Question

Extension of these results to the context of Gevrey classes?

Gevrey classes

For a real number $s \geq 1$, a function $f \in C^\infty(\Omega)$ is said to be Gevrey differentiable of order s at $x_0 \in \Omega$ if there exist a compact neighborhood K of x_0 and two constants $C, h > 0$ such that

$$\sup_{x \in K} \left| D^k f(x) \right| \leq C h^k (k!)^s, \quad \forall k \in \mathbb{N}_0.$$

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$\longrightarrow$ Intermediates between the space of analytic functions and the space of infinitely differentiable functions

A nowhere Gevrey differentiable function on a subset I of $\mathbb{R}$ is a function that is not Gevrey differentiable of order s at x_0, for any $x_0 \in I$ and any $s \geq 1$.

Existence of nowhere Gevrey differentiable functions

Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of $(0,+\infty)$ such that

$$\lambda_k \geq (k+1)^{(k+1)^2} \quad \text{and} \quad \lambda_{k+1} \geq 2\sum_{j=1}^{k} \lambda_j^{2+k-j}, \qquad \forall k \in \mathbb{N}.$$

The function f defined on $\mathbb{R}$ by

$$f(x) = \sum_{k=1}^{+\infty} \lambda_k^{1-k} e^{i\lambda_k x}$$

belongs to $\mathcal{C}^\infty(\mathbb{R})$ and is nowhere Gevrey differentiable on $\mathbb{R}$.

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Result (Bastin, E., Nicolay 2012; Bastin, Conejero, E., Seoane 2014)

The set of nowhere Gevrey differentiable functions is residual, prevalent and $\mathfrak{c}$-dense-algebrable in $C^\infty([0,1])$.

Lineability. If $e_\alpha(x) := \exp(\alpha x)$ and if f is any nowhere Gevrey differentiable function, then it suffices to take

$$\mathcal{D} = \operatorname{span}\{f e_\alpha : \alpha \in \mathbb{R}\}.$$

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Algebrability. For every $n \geq 2$, the function

$$f_n(x) := \exp\left(-x^{-\frac{1}{n-1}}\right)\chi_{(0,+\infty)}(x)$$

is Gevrey differentiable of order n on $\mathbb{R}$. We consider the function ψ_n defined by

$$\psi_n(x) := f_n(x) f_n(1-x)$$

and we define ρ by

$$\rho(x) := \sum_{n=2}^{+\infty} C_n \psi_n\left(2^n x - \lfloor 2^n x \rfloor\right).$$

Then, take the minimum algebra $\mathcal{A}$ which contains the family of the nowhere Gevrey functions ρe_α with $\alpha \in \mathcal{H}$, where $\mathcal{H}$ denote a Hamel basis of $\mathbb{R}$.

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Definition

Let Ω be an open subset of $\mathbb{R}$ and M be a weight sequence. The space $\mathcal{E}_{\{M\}}(\Omega)$ is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact } \exists h > 0 \text{ such that } \|f\|^M_{K,h} < +\infty \right\},$$

where

$$\|f\|^M_{K,h} := \sup_{n\in\mathbb{N}_0} \sup_{x\in K} \frac{|D^n f(x)|}{h^n M_n}.$$

If $f \in \mathcal{E}_{\{M\}}(\Omega)$, we say that f is M-ultradifferentiable of Roumieu type on Ω.

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Particular case. The weight sequences $(k!)_{k\in\mathbb{N}_0}$ and $((k!)^s)_{k\in\mathbb{N}_0}$ with $s > 1$.

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Let Ω be an open subset of $\mathbb{R}$ and M be a weight sequence. The space $\mathcal{E}_{(M)}(\Omega)$ is defined by

$$\mathcal{E}_{(M)}(\Omega) := \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall K \subseteq \Omega \text{ compact }, \forall h > 0,\ \|f\|_{K,h}^{M} < +\infty \right\}.$$

If $f \in \mathcal{E}_{(M)}(\Omega)$, we say that f is M-ultradifferentiable of Beurling type on Ω and we use the representation

$$\mathcal{E}_{(M)}(\Omega) = \underset{\overleftarrow{K \subseteq \Omega}}{\text{proj}}\ \underset{\overleftarrow{h>0}}{\text{proj}}\ \mathcal{E}_{M,h}(K)$$

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Questions.

- When do we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$?
- In that case, “how small” is $\mathcal{E}_{\{M\}}(\Omega)$ in $\mathcal{E}_{(N)}(\Omega)$?

General assumptions.

- We assume that any weight sequence M is logarithmically convex, i.e.

$$M_k^2 \leq M_{k-1}M_{k+1} \quad \forall k \in \mathbb{N}\,.$$

  It implies that the space $\mathcal{E}_{\{M\}}(\Omega)$ is an algebra.
- Since we have $\mathcal{E}_{[M]}(\Omega) = \mathcal{E}_{[M']}(\Omega)$ where $M'_k = \frac{M_k}{M_0}$ for every k, we assume that any weight sequence M is such that $M_0 = 1$.
- We usually assume that any weight sequence M is non-quasianalytic. Then given an open subset Ω of $\mathbb{R}$ and a compact $K \subseteq \Omega$, there exists a function of $\mathcal{E}_{\{M\}}(\mathbb{R})$ having a compact support included in Ω and being identically equal to 1 in K.

Let I be an open interval of $\mathbb{R}$. A class $\mathcal{E}_{\{M\}}(I)$ is quasianalytic if 0 is its unique function f for which there is a point $x \in I$ such that $D^n f(x) = 0$ for every $n \in \mathbb{N}_0$. If this is not the case, we say that the class $\mathcal{E}_{\{M\}}(I)$ is non-quasianalytic.

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Denjoy-Carleman Theorem (1921,1926)

Let M be a log-convex weight sequence and let I be an open interval of $\mathbb{R}$. TFAE:

1. $\mathcal{E}_{\{M\}}(I)$ is quasianalytic,
2. $\displaystyle\sum_{k=0}^{+\infty} \frac{1}{L_k} = +\infty$ where $L_k = \inf_{j \geq k} M_j^{1/j}$,
3. $\displaystyle\sum_{k=1}^{+\infty} \frac{M_{k-1}}{M_k} = +\infty$,
4. $\displaystyle\sum_{k=1}^{+\infty} (M_k)^{-1/k} = +\infty$.

If one of the equivalent conditions is satisfied, we say that the weight sequence M is quasianalytic. Otherwise, we say that the sequence is non-quasianalytic.

Inclusions between Denjoy-Carleman classes

Definition

Given two weight sequences M and N, we write

$$M \lhd N \iff \lim_{k\to+\infty}\left(\frac{M_k}{N_k}\right)^{\frac{1}{k}} = 0$$
$$\iff \forall\rho > 0\ \exists C > 0\ :\ M_k \leq C\rho^k N_k\ \ \forall k \in \mathbb{N}_0\,.$$

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$$\iff \forall \rho > 0 \; \exists C > 0 \; : \; M_k \leq C \rho^k N_k \;\; \forall k \in \mathbb{N}_0 \, .$$

Proposition

Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then

$$M \lhd N \iff \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$$

and in this case, the inclusion is strict.

Proof.

$\Rightarrow$ Ok. $\sup_{x \in K} |D^k f(x)| \leq C h^k M_k \leq C'(h\rho)^k N_k$

$\Leftarrow$??

Lemma 1

Let M and N be two weight sequences such that $M \lhd N$. Then there exist a weight sequence L such that

$$M \lhd L \lhd N.$$

Idea.

It suffices to set

$$L_k = \sqrt{M_k N_k}, \quad \forall k \in \mathbb{N}_0 \,.$$

Then

$$\left(\frac{M_k}{L_k}\right)^{\frac{1}{k}} = \left(\frac{M_k}{N_k}\right)^{\frac{1}{2k}} \to 0$$

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Lemma 2

Let M be a weight sequence and θ be the function defined on $\mathbb{R}$ by

$$\theta(x) = \sum_{k=1}^{+\infty} \frac{M_k}{2^k} \left(\frac{M_{k-1}}{M_k} \right)^k \exp\left(2i \frac{M_k}{M_{k-1}} x \right).$$

Then $\theta \in \mathcal{E}_{\{M\}}(\mathbb{R})$ and $|D^j \theta(0)| \geq M_j$ for all $j \in \mathbb{N}_0$. In particular, this function belongs to $\mathcal{E}_{\{M\}}(\mathbb{R}) \backslash \mathcal{E}_{(M)}(\mathbb{R})$.

Proposition

Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then

$$M \lhd N \Longleftrightarrow \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$$

and in this case, the inclusion is strict.

Proof.

$\Rightarrow$ Ok

$\Leftarrow$ Up to a translation, we can assume that $0 \in \Omega$. Lemma 2 gives a function $\theta \in \mathcal{E}_{\{M\}}(\Omega)$ such that $|D^k\theta(0)| \geq M_k$ for every $k \in \mathbb{N}_0$. Then, $\theta \in \mathcal{E}_{(N)}(\Omega)$ and for every $\rho > 0$, there is $C > 0$ such that

$$M_k \leq |D^k\theta(0)| \leq C\rho^k N_k, \quad \forall k \in \mathbb{N}_0\,.$$

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Strict inclusion. By Lemma 1, there exists a weight sequence L such that $M \lhd L \lhd N$. Again, Lemma 2 gives a function f which belongs to $\mathcal{E}_{\{L\}}(\Omega)$ but not to $\mathcal{E}_{(L)}(\Omega)$. Since $L \lhd N$, we have that $\mathcal{E}_{\{L\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$ and since $M \lhd L$, we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(L)}(\Omega)$. So $f \in \mathcal{E}_{(N)}(\Omega) \setminus \mathcal{E}_{\{M\}}(\Omega)$.

Construction

Definition

We say that a function is nowhere in $\mathcal{E}_{\{M\}}$ if its restriction to any open and non-empty subset Ω of $\mathbb{R}$ never belongs to $\mathcal{E}_{\{M\}}(\Omega)$.

Proposition

Assume that $M \lhd N$. If M is non-quasianalytic, there exists a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.

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Proof. From Lemma 1, there is $N^\star$ such that $M \lhd N^\star \lhd N$. Applying recursively this lemma, we get a sequence $(L^{(p)})_{p\in\mathbb{N}}$ of weight sequences such that

$$M \lhd L^{(1)} \lhd L^{(2)} \lhd \cdots \lhd L^{(p)} \lhd \cdots \lhd N^\star \lhd N.$$

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For every $p \in \mathbb{N}$, Lemma 2 allows us to consider a function $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ such that

$$|D^j f_p(0)| \geq L_j^{(p)}, \quad \forall j \in \mathbb{N}_0\,.$$

Since M is non-quasianalytic, there is $\phi \in \mathcal{E}_{\{M\}}(\mathbb{R})$ with compact support and identically equal to 1 in a neighborhood of the origin. Let $\{x_p : p \in \mathbb{N}_0\}$ be a dense subset of $\mathbb{R}$ with $x_0 = 0$. For every $p \in \mathbb{N}$, we can find $k_p > 0$ such that the function

$$\phi_p := \phi\big(k_p(\cdot - x_p)\big)$$

has its support disjoint from $\{x_0, \ldots, x_{p-1}\}$. We introduce the function g_p defined on $\mathbb{R}$ by

$$g_p(x) := \underbrace{f_p(x - x_p)}_{\in \mathcal{E}_{\{L(p)\}}(\mathbb{R})} \; \underbrace{\phi_p(x)}_{\in \mathcal{E}_{\{M\}}(\mathbb{R})} \quad \in \mathcal{E}_{(N^\star)}(\mathbb{R}).$$

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Let $\gamma_p > 0$ be such that

$$\sup_{x \in \mathbb{R}} |D^j g_p(x)| \leq \gamma_p N_j^\star, \quad \forall j \in \mathbb{N}_0$$

and define the function g by

$$g := \sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} g_p.$$

1. $g \in \mathcal{E}_{(N)}(\mathbb{R})$: for every $j \in \mathbb{N}_0$ and every $x \in \mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} |D^j g_p(x)| \leq \sum_{p=1}^{+\infty} \frac{1}{2^p} N_j^\star \leq N_j^\star$$

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2. g is nowhere in $\mathcal{E}_{\{M\}}$: By contradiction, assume that there exists an open subset Ω of $\mathbb{R}$ such that $g \in \mathcal{E}_{\{M\}}(\Omega)$. Let $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$. Remark that

$$\sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p = \underbrace{g}_{\in \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(L^{(p_0)})}(\Omega)} - \underbrace{\sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p}_{\in \mathcal{E}_{(L^{(p_0)})}(\Omega)} \in \mathcal{E}_{(L^{(p_0)})}(\Omega).$$

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But, since the support of g_p is disjoint of x_{p_0} for every $p > p_0$, we also have

$$\left| \sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} D^j g_p(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| D^j g_{p_0}(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| D^j f_{p_0}(0) \right| \geq \frac{1}{\gamma_{p_0} 2^{p_0}} L_j^{(p_0)}$$

for every $j \in \mathbb{N}_0$, hence a contradiction.

Remark. Given a sequence $(L^{(p)})_{p\in\mathbb{N}}$ such that

$$M \lhd L^{(1)} \lhd L^{(2)} \lhd \cdots \lhd L^{(p)} \lhd \cdots \lhd N^* \lhd N$$

and a dense subset $\{x_p : p \in \mathbb{N}_0\}$ of $\mathbb{R}$, we have constructed a function $g \in \mathcal{E}_{\{N^*\}}(\mathbb{R})$ which is not in $\mathcal{E}_{(L^{(p)})}(\Omega)$ for every neighbourhood Ω of x_p.

Lineability

Maximal lineability

Assume that $M \rhd N$. If M is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is $\mathfrak{c}$-dense-lineable.

Proof. For every $t \in (0,1)$, we set

$$L_k^{(t)} := (M_k)^{1-t}(N_k)^t \quad \forall k \in \mathbb{N}_0 .$$

Then $M \rhd L^{(t)} \rhd N$ for all $t \in (0,1)$ and $L^{(t)} \rhd L^{(s)}$ if $t < s$.

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Then $M \rhd L^{(t)} \rhd N$ for all $t \in (0,1)$ and $L^{(t)} \rhd L^{(s)}$ if $t < s$.
Remark that

$$M \rhd L^{(\frac{t}{2})} \rhd L^{(\frac{2t}{3})} \rhd L^{(\frac{3t}{4})} \rhd \cdots \rhd L^{(t)} \rhd N, \quad \forall t \in (0,1).$$

and we can consider $g_t \in \mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$ which is not in $\mathcal{E}_{\left(L^{\left((1-\frac{1}{p})t\right)}\right)}(\Omega)$, for any open neighbourhood Ω of x_p and for any $p \geq 2$.

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Let us fix $\alpha_1, \dots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \cdots < t_N$ in $(0,1)$, and let us consider the function

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Assume that there exists an open subset Ω of $\mathbb{R}$ such that $G \in \mathcal{E}_{\{M\}}(\Omega)$. We fix $p \in \mathbb{N}$ such that $x_p \in \Omega$ and $t_{N-1} < \left(1 - \frac{1}{p}\right) t_N$. Again, the function g_{t_n} belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})$ for every $n \leq N-1$ and it follows that the function

$$g_{t_N} = \frac{1}{\alpha_N}\left(G - \sum_{n=1}^{N-1} \alpha_n g_{t_n}\right)$$

belongs to $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega)$. From the choice of p, we have

$$\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega) \subset \mathcal{E}_{\left(L^{\left((1-\frac{1}{p_0})t_N\right)}\right)}(\Omega)$$

and this leads to a contradiction.

2. $\dim \mathcal{D} = \mathfrak{c}$
Assume there exist $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \cdots < t_N$ in $(0,1)$ such that $\sum_{n=1}^{N} \alpha_n g_{t_n} = 0$. Then

$$g_{t_N} = \frac{-1}{\alpha_N} \sum_{n=1}^{N-1} \alpha_n g_{t_n}$$

and since $g_{t_n} \in \mathcal{E}_{\{L^{(t_n)}\}}(\mathbb{R}) \subset \mathcal{E}_{\left\{L^{(t_{N-1})}\right\}}(\mathbb{R})$ for every $n \leq N-1$, we get that

$$g_{t_N} \in \mathcal{E}_{\left\{L^{(t_{N-1})}\right\}}(\mathbb{R}) \subset \mathcal{E}_{\left(L^{((1-\frac{1}{p_0})t_N)}\right)}(\mathbb{R})$$

if p is such that $(1-\frac{1}{p})t_N > t_{N-1}$. This is a contradiction.

Dense-lineability

Remark. The set of polynomials is dense in $\mathcal{E}_{(N)}(\mathbb{R})$. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence of different elements of $(0,1)$ and let $(P_{t_m})_{m\in\mathbb{N}}$ be a dense sequence of polynomials in $\mathcal{E}_{(N)}(\mathbb{R})$.

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We choose for every $m \in \mathbb{N}$ a positive constant k_m such that $k_m g_{t_m} \in U_m$, where $\{U_m : m \in \mathbb{N}\}$ is a basis of convex balanced absorbing neighbourhoods of 0 in $\mathcal{E}_{(N)}(\mathbb{R})$.

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We consider the linear space $\mathcal{D}_d$ spanned by

$$\left\{P_t + k_t g_t : t \in (0,1)\right\}$$

where $k_t = 1$ and $P_t = 0$ if $t \neq t_m$ for every $m \in \mathbb{N}$.

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Result

Assume that $M \rhd N$. If M is non quasianalytic, then $\mathcal{D}_d$ is dense in $\mathcal{E}_{(N)}(\mathbb{R})$, $\dim \mathcal{D}_d = \mathfrak{c}$ and any non zero function of $\mathcal{D}_d$ is nowhere in $\mathcal{E}_{\{M\}}$. In particular, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.

Case of countable unions

Let N be a weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \triangleright N$ for every $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is prevalent, residual and $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{(N)}(\mathbb{R})$.

Idea. Construct a weight sequence P such that

$$\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}} \subseteq \mathcal{E}_{\{P\}} \subsetneq \mathcal{E}_{(N)}.$$

Gevrey classes

They correspond to Roumieu classes given by the weight sequence

$$M_k := (k!)^\alpha, \quad k \in \mathbb{N}_0 \,.$$

Particular case of Gevrey classes

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$, is $\mathfrak{c}$-dense-lineable in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

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Proof. It suffices to take the weight sequences $M^{(n)}$ ($n \in \mathbb{N}$) given by

$$M_k^{(n)} := (k!)^{\beta_n}, \quad k \in \mathbb{N}_0,$$

where $(\beta_n)_{n\in\mathbb{N}}$ is an increasing sequence of $(1, \alpha)$ that converges to α.

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where $(\beta_n)_{n\in\mathbb{N}}$ is an increasing sequence of $(1, \alpha)$ that converges to α.

Proposition (Schmets, Valdivia 1991)

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$ is residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

More with weight functions

Definition

A function $\omega : [0, +\infty[\to [0, +\infty[$ is called a weight function if it is continuous, increasing and satisfies $\omega(0) = 0$ as well as the following conditions

(α) There exists $L \geq 1$ such that $\omega(2t) \leq L\omega(t) + L,\ t \geq 0$,

(β) $\displaystyle\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$,

(γ) $\log(t) = o(\omega(t))$ as t tends to infinity,

(δ) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex on $[0, +\infty[$.

The Young conjugate of φ_ω is defined by

$$\varphi_\omega^*(x) := \sup\{xy - \varphi_\omega(y) : y > 0\}, \quad x \geq 0.$$

Examples.

$\omega(t) = t \log(1+t)^{-\alpha}$, $\alpha > 1$

$\omega(t) = t^\alpha$, $0 < \alpha < 1$ (Gevrey classes)

For a compact subset K of $\mathbb{R}^n$ and $m \in \mathbb{N}$, we define the space $\mathcal{E}_\omega^m(K)$ as the space of functions $f \in \mathcal{E}(K)$ such that

$$\|f\|_{K,m} := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp\left(-\frac{1}{m}\varphi_\omega^*(m|\alpha|)\right) < +\infty.$$

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Definition

If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, we define the space $\mathcal{E}_{\{\omega\}}(\Omega)$ of ω-ultradifferentiable functions of Roumieu type on Ω by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \{f \in \mathcal{E}(\Omega) : \forall K \subset \Omega \text{ compact } \exists m \in \mathbb{N} \text{ such that } \|f\|_{K,m} < +\infty\}.$$

Definition

If ω is a weight function and if Ω is an open subset of $\mathbb{R}^n$, the space $\mathcal{E}_{(\omega)}(\Omega)$ of ω-ultradifferentiable functions of Beurling type on Ω is defined by

$$\mathcal{E}_{(\omega)}(\Omega) := \{f \in \mathcal{E}(\Omega) : \forall K \subset \Omega \text{ compact}, \forall m \in \mathbb{N},\ p_{K,m}(f) < +\infty\},$$

where for every compact subset K of $\mathbb{R}^n$ and every $m \in \mathbb{N}$

$$p_{K,m}(f) := \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

We endow the space $\mathcal{E}_{(\omega)}(\Omega)$ with its natural Fréchet space topology.

Given two weight functions ω and σ, we write

$$\omega \lhd \sigma \Longleftrightarrow \sigma(t) = o(\omega(t)) \text{ as } t \to +\infty.$$

Result

Let ω and σ be two weight functions such that $\omega \lhd \sigma$. If Ω is a convex open subset of $\mathbb{R}^n$, then $\mathcal{E}_{\{\omega\}}(\Omega)$ is strictly included in $\mathcal{E}_{(\sigma)}(\Omega)$.

Result

Let ω and σ be two weight functions such that $\omega \lhd \sigma$. The set of functions of $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$ which are nowhere in $\mathcal{E}_{\{\omega\}}$ is dense-lineable in $\mathcal{E}_{(\sigma)}(\mathbb{R}^n)$.