A CHARACTERIZATION OF THE RADON-NIKODYM PROPERTY.

APPLICATION TO THE CONSTRUCTION OF ALMOST CLASSICAL SOLUTIONS OF HAMILTON-JACOBI EQUATIONS.

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Introduction.

(Whitney) if $d \geq 2$, there exists $u : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{C}^1$ and $\gamma : [0,1] \to \mathbb{R}^d$ continuous such that $u\Big(\gamma(0)\Big) \neq u\Big(\gamma(1)\Big)$ and $Du\Big(\gamma(t)\Big) = 0$ for all $t \in [0,1]$.

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Moreover if $\varepsilon > 0$ and $a \in \mathbb{R}^d$, $\|a\| = 1$ are fixed,
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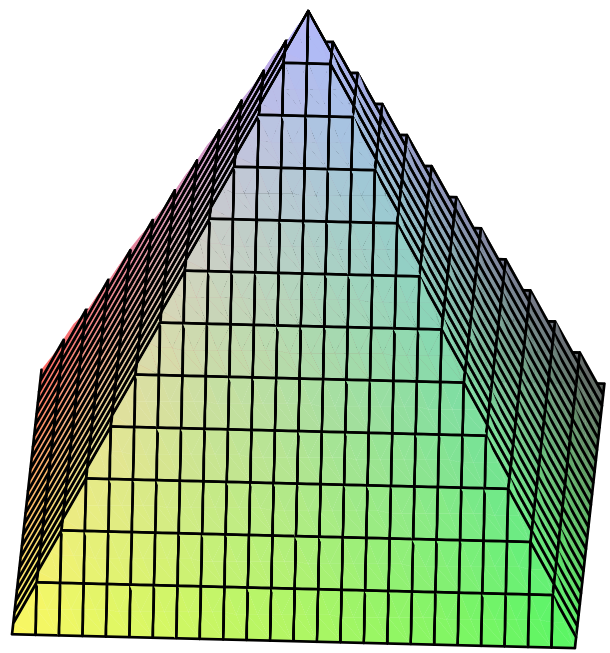
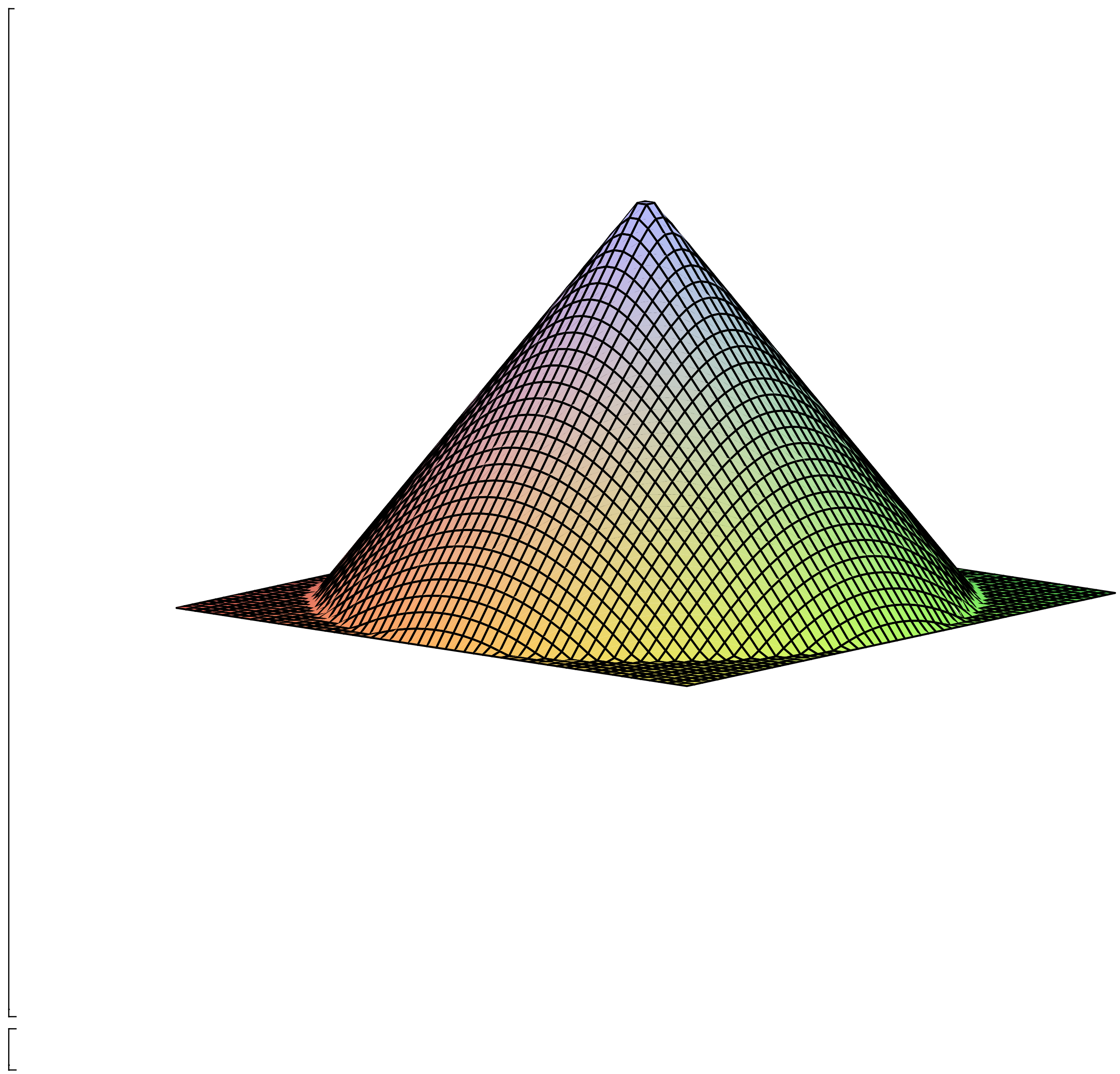
The function $u(x) = d(x, \partial\Omega)$ is the viscosity solution of $\|Du(x)\| = 1$ on Ω with the boundary condition $u(x) = 0$ if $x \in \partial\Omega$, but is not differentiable on Ω.





Construction of $u : \mathbb{R}^d \to \mathbb{R}$ differentiable at each point, such that $u(x) = 0$ si $x \notin \Omega$ and $\|Du(x)\| = 1$ a. e. on Ω.

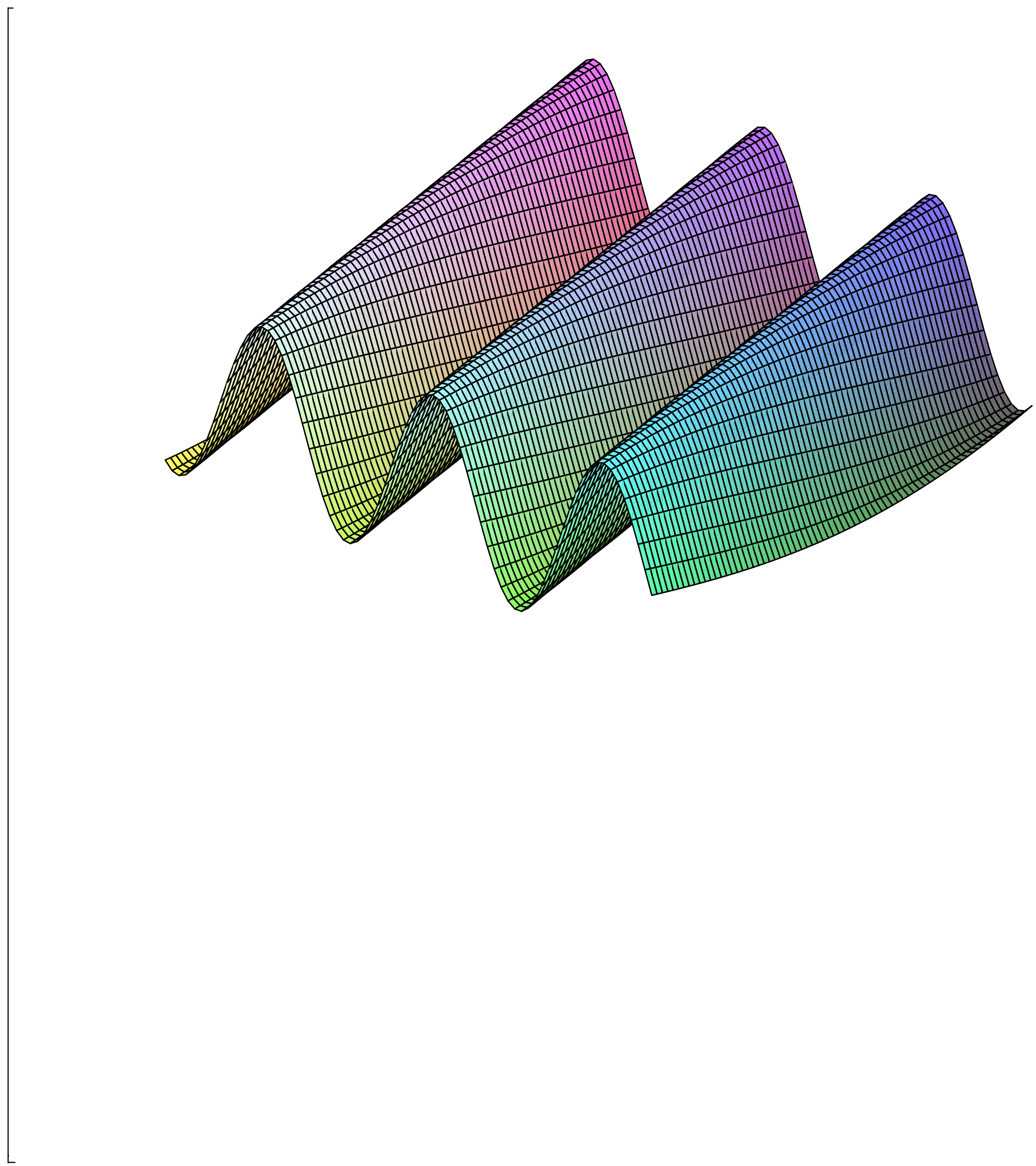
Lemma. Let $a \in \mathbb{R}^d \backslash \{0\}$, Q be a cube of $\mathbb{R}^d$, and $\varepsilon > 0$.

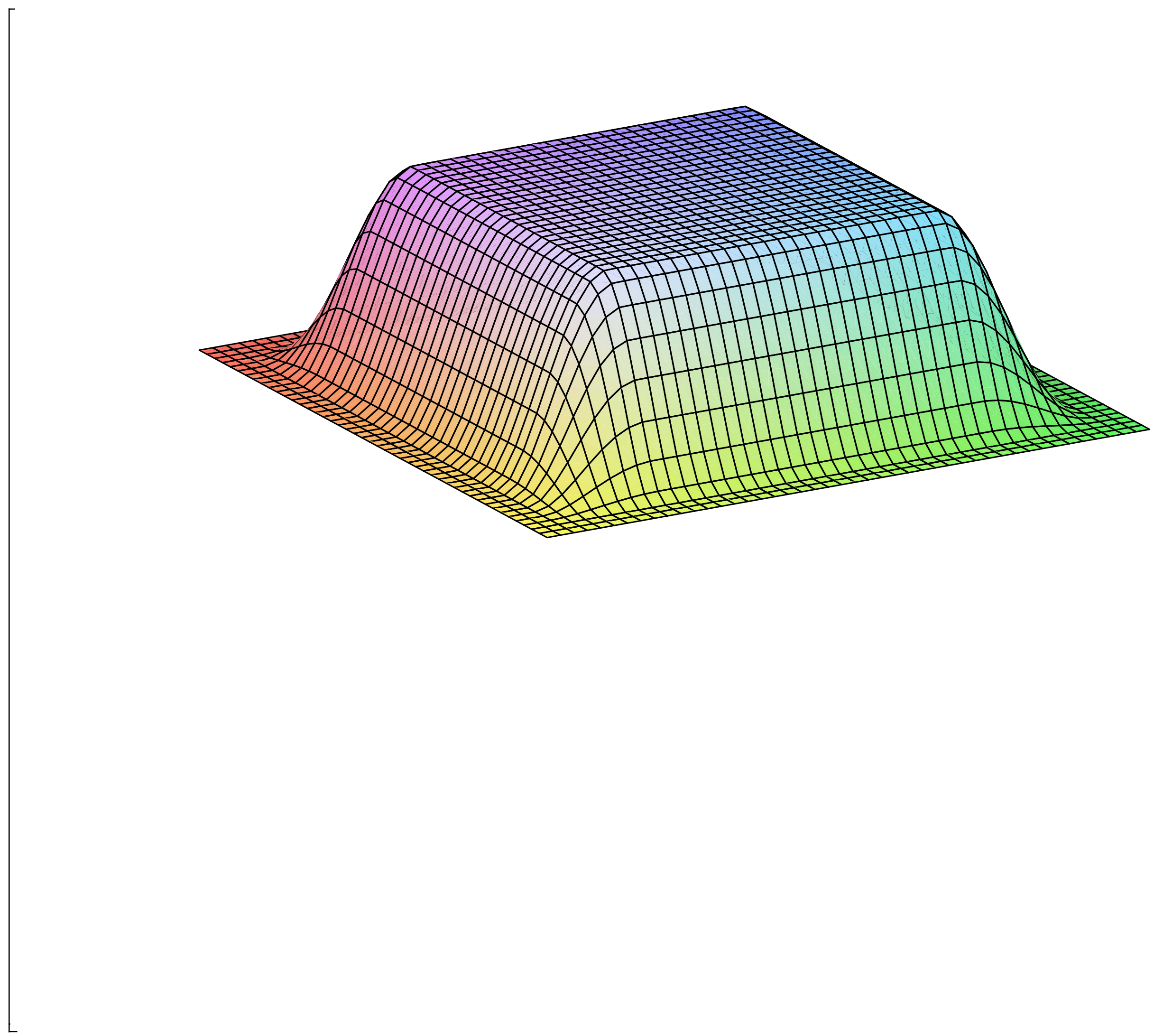
Then, $\exists u : \mathbb{R}^d \to \mathbb{R}$ bounded, of class $\mathcal{C}^\infty$, such that :

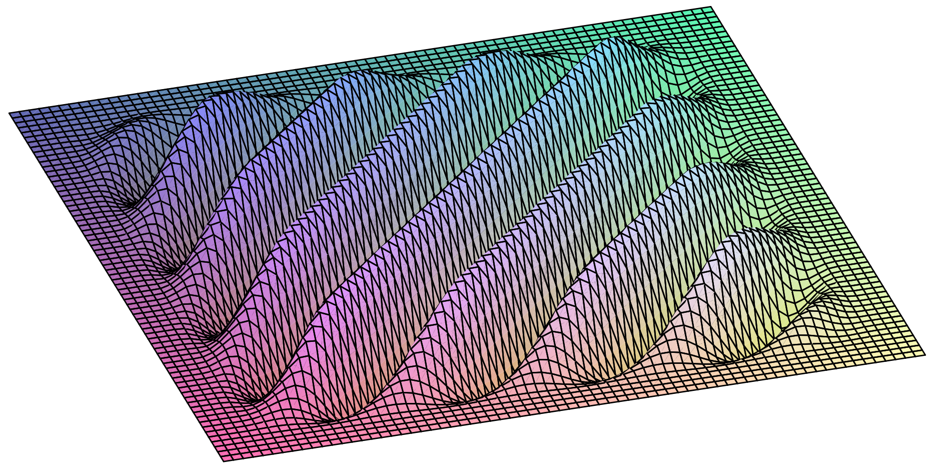
(a) $u \equiv 0$ in a neighbourhood of ∂Q and $\|u\|_\infty \leq \varepsilon$.

(b) $\lambda_d\Big(\{x \in Q;\ Du(x) = -a \text{ or } Du(x) = a\}\Big) \geq (1 - \varepsilon)\lambda_d(Q)$.

(c) $Du = v + w$ with $\|w\|_\infty < \varepsilon$,
$\{v(x);\ x \in Q\} \subset [-a, a]$ and v piecewise constant on Q.







Construction of $u : \mathbb{R}^d \to \mathbb{R}$ differentiable at each point, such that $u(x) = 0$ si $x \notin \Omega$ and $\|Du(x)\| = 1$ a. e. on Ω.

Lemma. Let $a \in \mathbb{R}^d \backslash \{0\}$, Q be a cube of $\mathbb{R}^d$, and $\varepsilon > 0$. Then, $\exists u : \mathbb{R}^d \to \mathbb{R}$ bounded, of class $\mathcal{C}^\infty$, such that :
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For each n, $\mathcal{Q}_n$ is a "partition" of $[0,1]^d$ into cubes and $\mathcal{Q}_{n+1}$ is a refinement of $\mathcal{Q}_n$.

$u_n \in \mathcal{C}^\infty(\mathbb{R}^d)$, such that $\forall Q \in \mathcal{Q}_n$, $u_{n|Q}$ defined using the lemma, with $a = a(Q)$ et $\varepsilon = \varepsilon_n$ to be chosen.

solution : $u = \sum_{n=0}^{\infty} u_n$

Differentiability criterium.

X, Y Banach spaces, $u_n : X \to Y$, $n \geq 1$, $\mathcal{C}^1$ such that :

(1) For all $x \in X$, $\left(\sum Du_n(x)\right)$ converges.

(2) $\left(Du_n\right)$ converges uniformly to 0.

(3) $\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)$.

(4) $\lim_{n\to\infty} osc\left(\sum_{k=1}^{n} Du_k, \|u_{n+1}\|_\infty\right) = 0$.

Then $u := \sum_{n=1}^{\infty} u_n$ is well defined, everywhere differentiable, and $Du(x) = \sum_{n=1}^{\infty} Du_n(x)$ for all $x \in X$.

Recall : $osc(f, \delta) := \sup\left\{||f(x) - f(y)||;\ ||x - y|| < \delta\right\}$.

How to ensure condition (1) together with the fact that $||Du(x)|| = ||\sum_{n=1}^{\infty} Du_n(x)|| = 1$ for almost every $x \in \Omega$?

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Theorem : $\exists t : \mathbb{R}^d \to S_{\mathbb{R}^d}$ such that if $\{a_n; n \in \mathbb{N}\} \subset \mathbb{R}^d$ is a bounded sequence satisfying $\langle t(a_n), a_{n+1} - a_n\rangle \geq 0$ for all n, then (a_n) converges.

This last theorem involves a monotony condition.

So we are led to the following question :

Is it possible to extend the assertion
Each non increasing bounded below sequence converges
in a Banach space setting ?

Yes if X has the Radon-Nikodym property.

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The Radon-Nikodym property.

Definition. Let X be a Banach space. X has the Radon-Nikodym property if, whenever C is a closed convex bounded subset of X and $\eta > 0$, there exists $g \in X^*$ and $c \in \mathbb{R}$ such that

$$C \cap \{g < c\} \neq \emptyset \qquad \text{and} \qquad diam\Big(C \cap \{g < c\}\Big) < \varepsilon.$$

Examples. X reflexive or X separable dual space
$\Rightarrow$ X has RNP.

In particular, L^p spaces, $(1 < p < +\infty)$ and ℓ^1 have RNP.

But $L^1([0,1])$ and $\mathcal{C}(K)$ spaces (K infinite compact) fail RNP.

Known characterizations.

Theorem. Let X be a Banach space. T.F.A.E. :

(1) X has the Radon-Nikodym property.

(2) Each X-valued measure on $[0,1]$ which is absolutly continuous w. r. t. Lebesgue measure has a density.

(3) $L^1([0,1],X)^* = L^\infty([0,1],X^*)$.

(4) If (X_n) is a martingale with values in B_X, then (X_n) converges a. s..

(5) If $f:\mathbb{R}\to X$ is Lipschitz, then f is differentiable a. e. (at least at one point).

(6) If C is a closed convex bounded subset of X, and if $f:C\to\mathbb{R}$ is $\ell.s.c.$ and bounded below, then $\{g\in X^*;\ f+g$ has a strong min. on $C\}$ is dense in X^*.

The main result (with O. Madiedo).

Theorem : If X is a Banach space, T.F.A.E. :

(1) X has the Radon-Nikodym property.

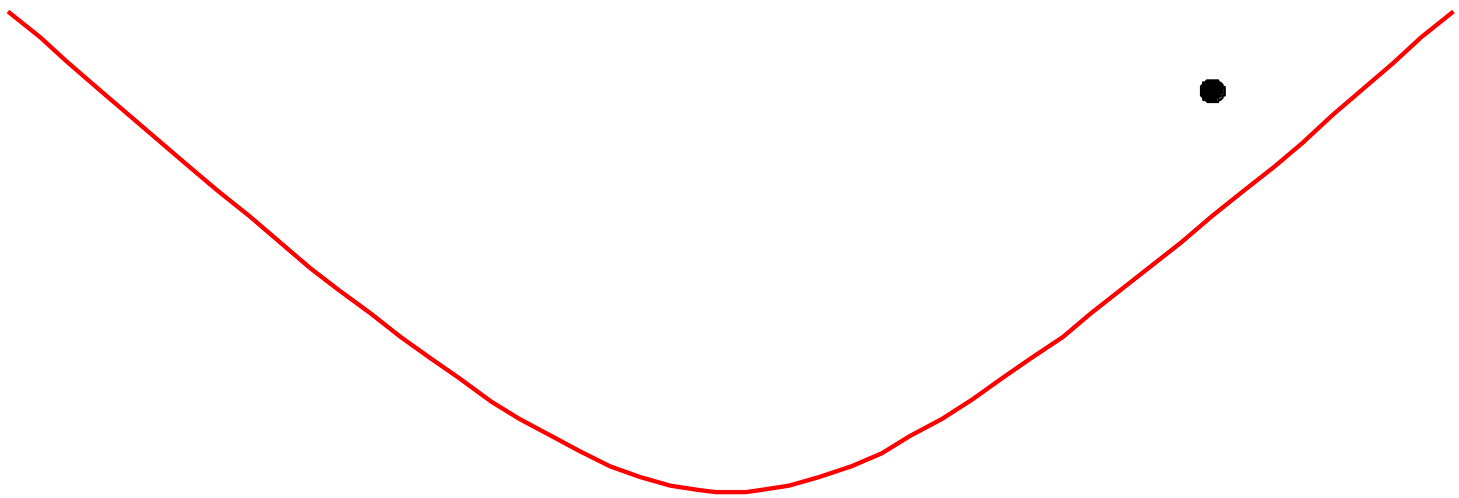
(2) For all $f \in S_{X^*}$ and all $\varepsilon > 0$,
there exists $t : X \to S_{X^*} \cap B(f, \varepsilon)$
such that for all sequence (x_n) in X,

if $\left(f(x_n) - \varepsilon \|x_n\| \right)$ is bounded below
and if $\langle\, t(x_n), x_{n+1} - x_n \,\rangle \leq 0$ for all n,

then (x_n) converges.

This result is non trivial even when $dim(X) = 2$.

Interpretation with games.



If $p \in \mathbb{R}$, we define $\Lambda_p = \{x \in X : f(x) \geq \varepsilon\|x\| + p\}$.

Player 1 chooses $x_n \in \Lambda_p$. $\Big(f(x_n) - \varepsilon\|x_n\|\Big)$ bounded below
Player 2 chooses slices S_n of Λ_p.
Player 1 start the game and chooses $x_1 \in \Lambda_p$.

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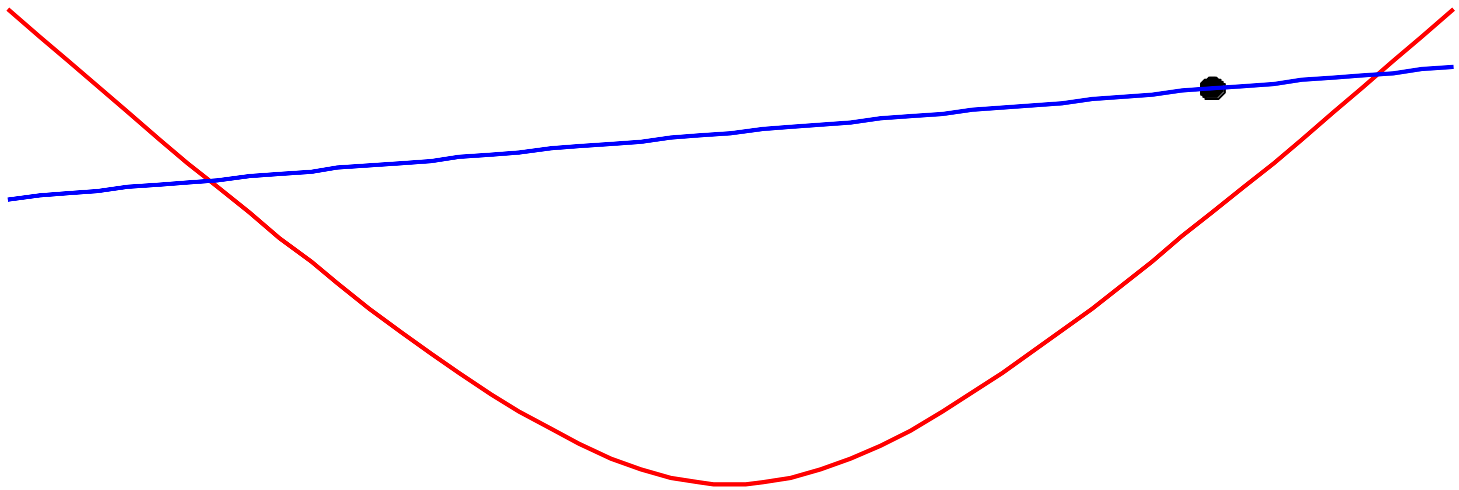

Player 2 then chooses a slice $S_1 = \{x \in \Lambda_p;\ f_1(x) \leq f_1(x_1)\}$.

$t(x_1) = f_1$

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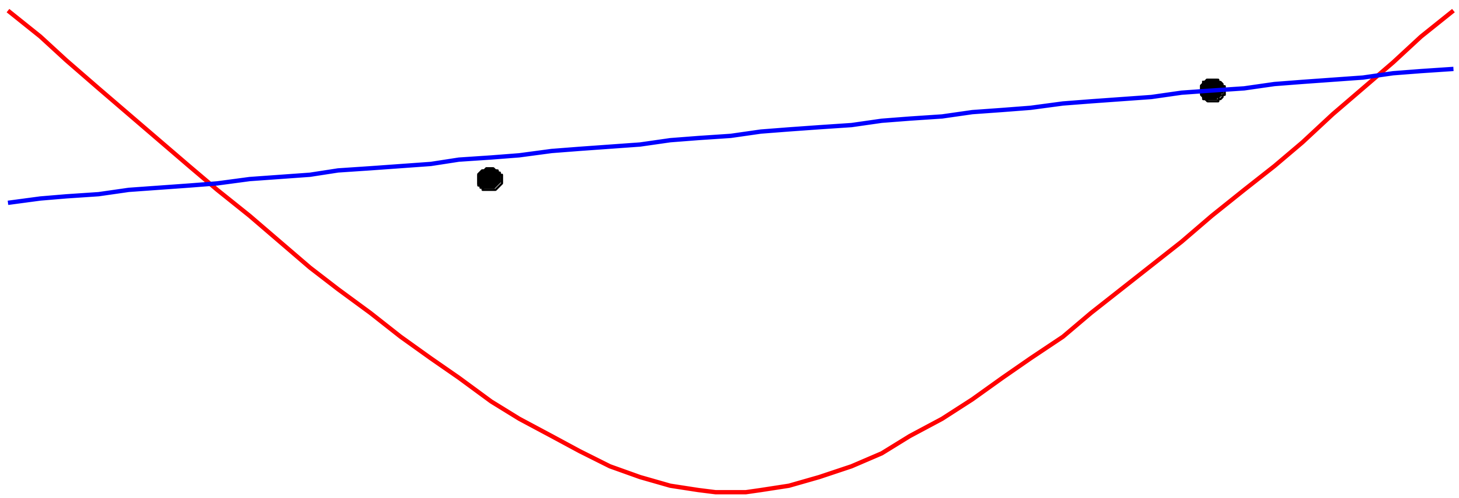

Player 1 chooses a point $x_2 \in S_1$.

Hypothesis $\langle\, t(x_n), x_{n+1} - x_n \,\rangle \leq 0$

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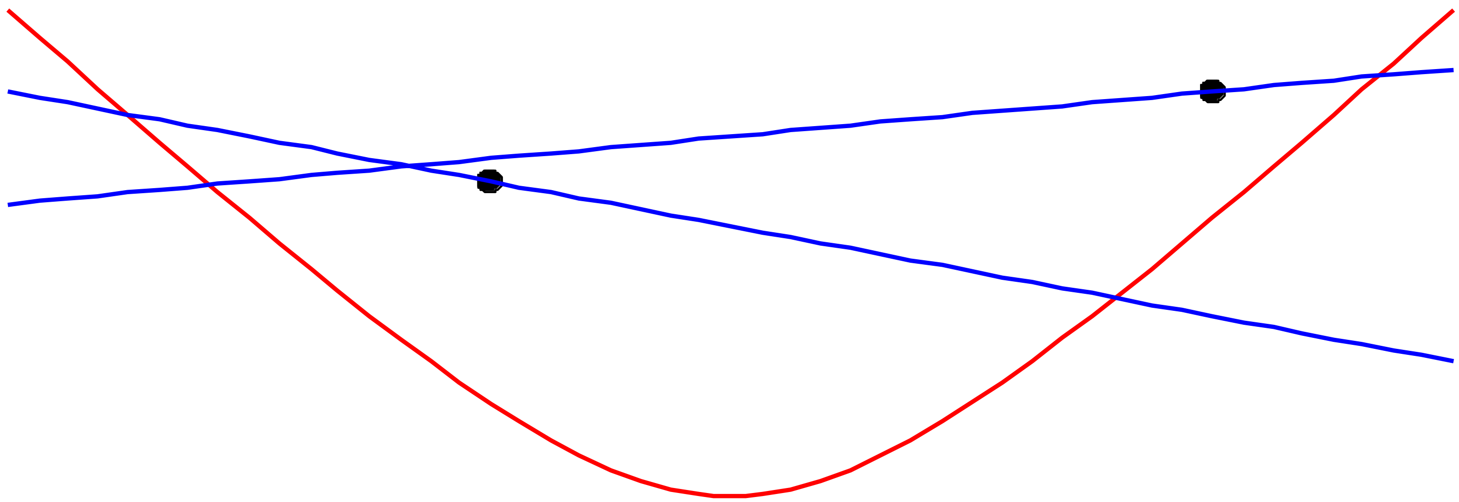

Player 2 chooses a slice $S_2 = \{x \in \Lambda_p;\ f_2(x) \leq f_2(x_2)\}$.

The slice S_2 is not necessarily included in S_1.

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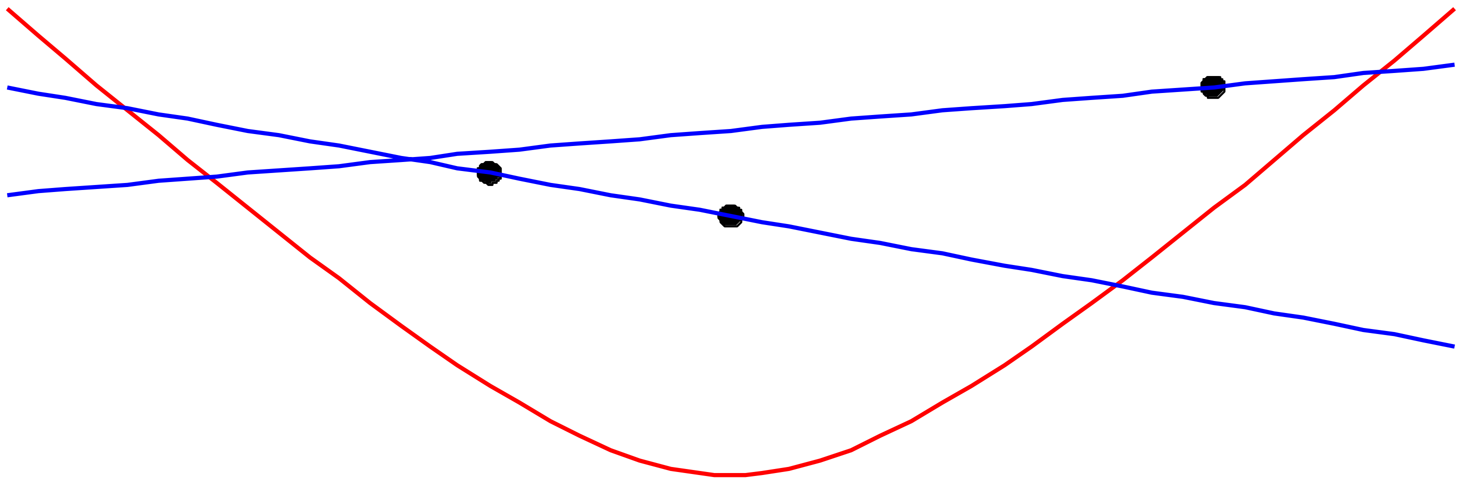
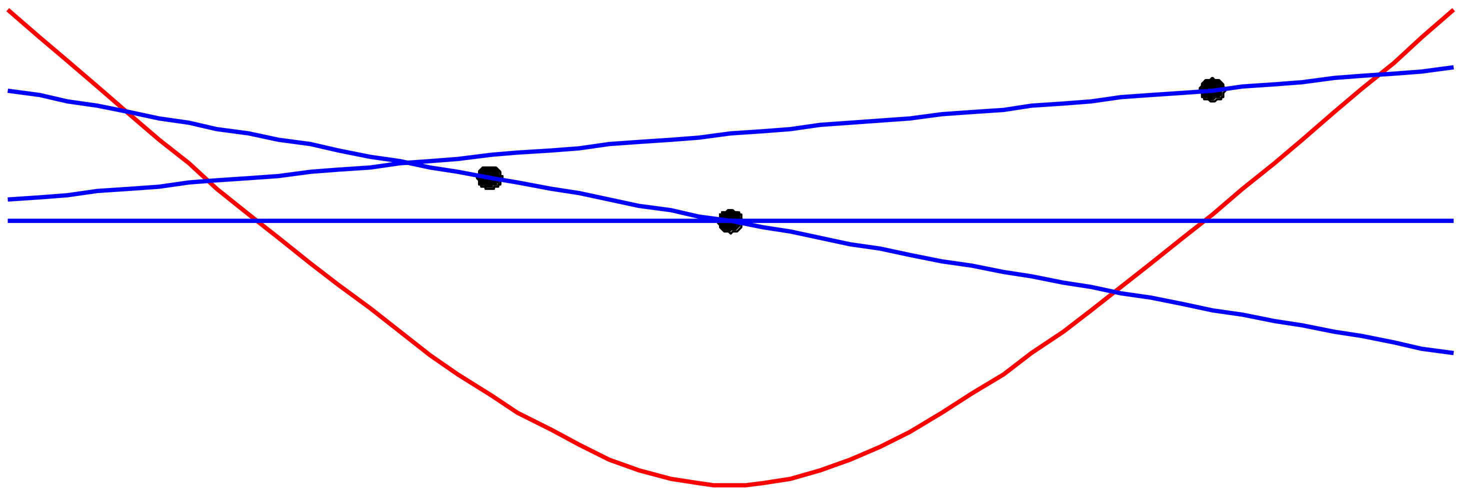


Player 1 chooses a point $x_3 \in S_2$.

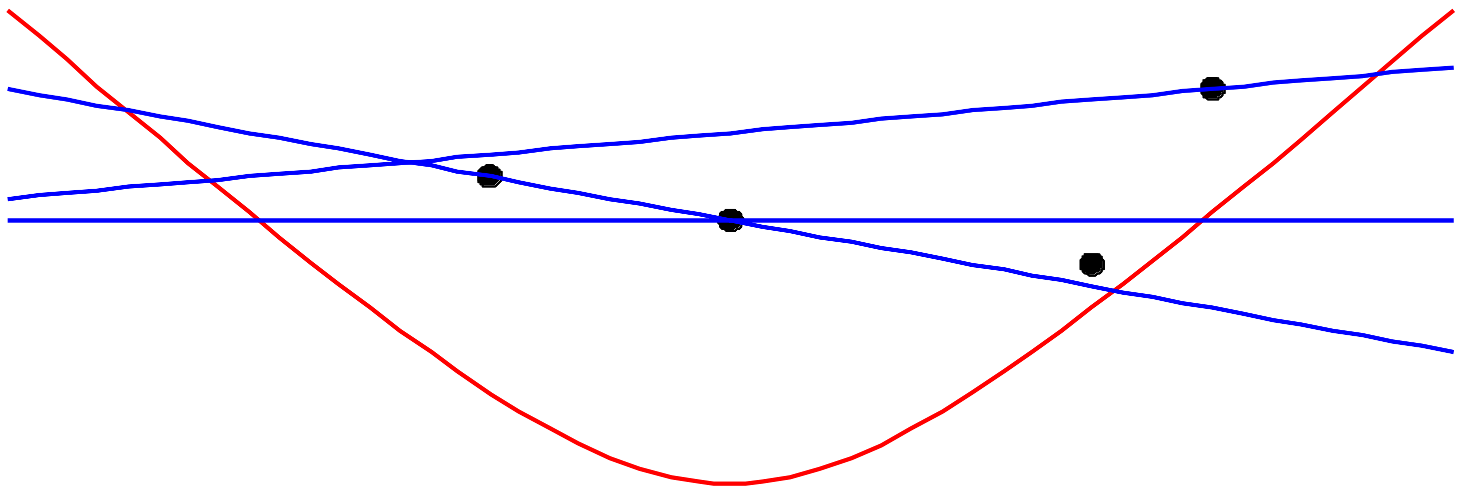
Interpretation with games.



And so on. Player 1 constructs a sequence (x_n) in $\Lambda_p \subset X$.
And player 2 constructs a sequence (f_n) in X^*,
defining slices S_n of Λ_p.

Player 1 is a thief and player 2 is a policeman.
Player 2 (the policeman) wishes that the sequence (x_n) converges.

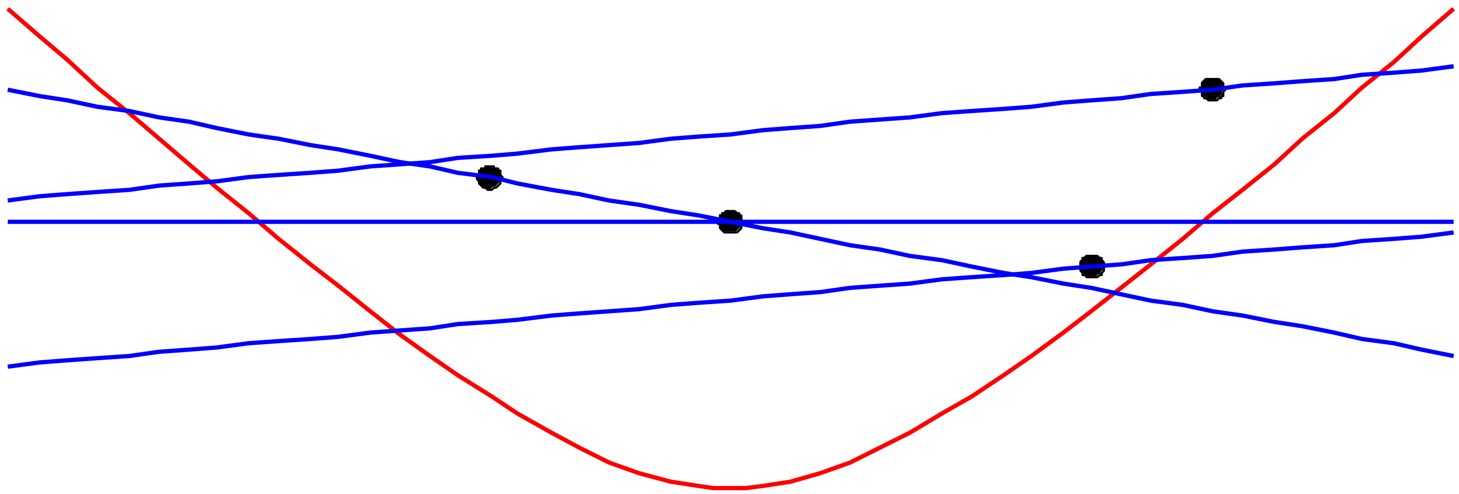
Interpretation with games.



Player 2 (the thief) wishes to escape.

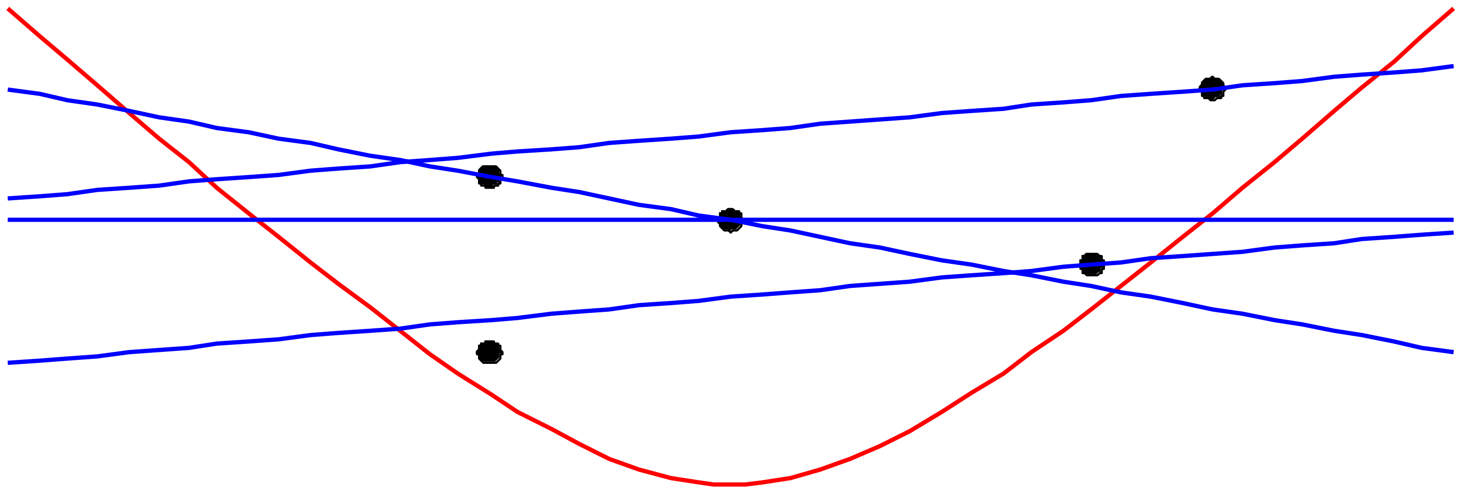
i. e. player 2 wins if the sequence (x_n) diverges.

Interpretation with games.



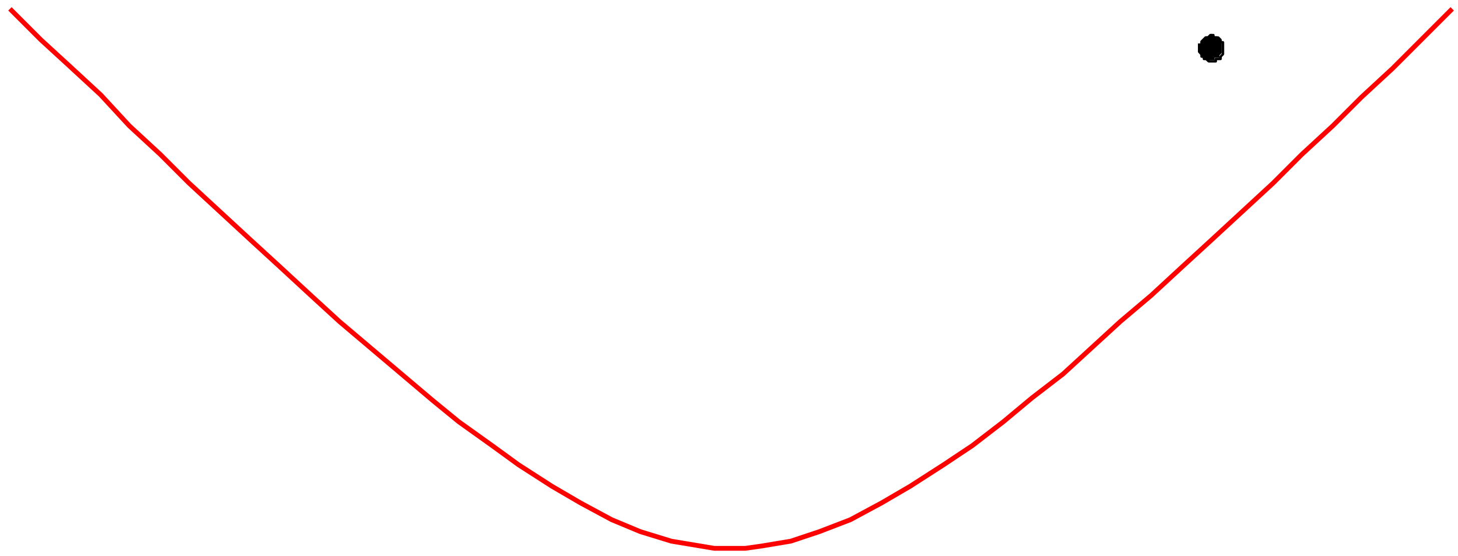
A winning tactic for the policeman is a choice of slices depending only on the last position of the thief, that guaranties that the sequence (x_n) converges.

Interpretation with games.



The policeman has a winning tactic if and only if the space X where the thief lives has RNP.

The constant tactic $t(x) = f$ for all $x \in X$ is not a winning tactic for the policeman.



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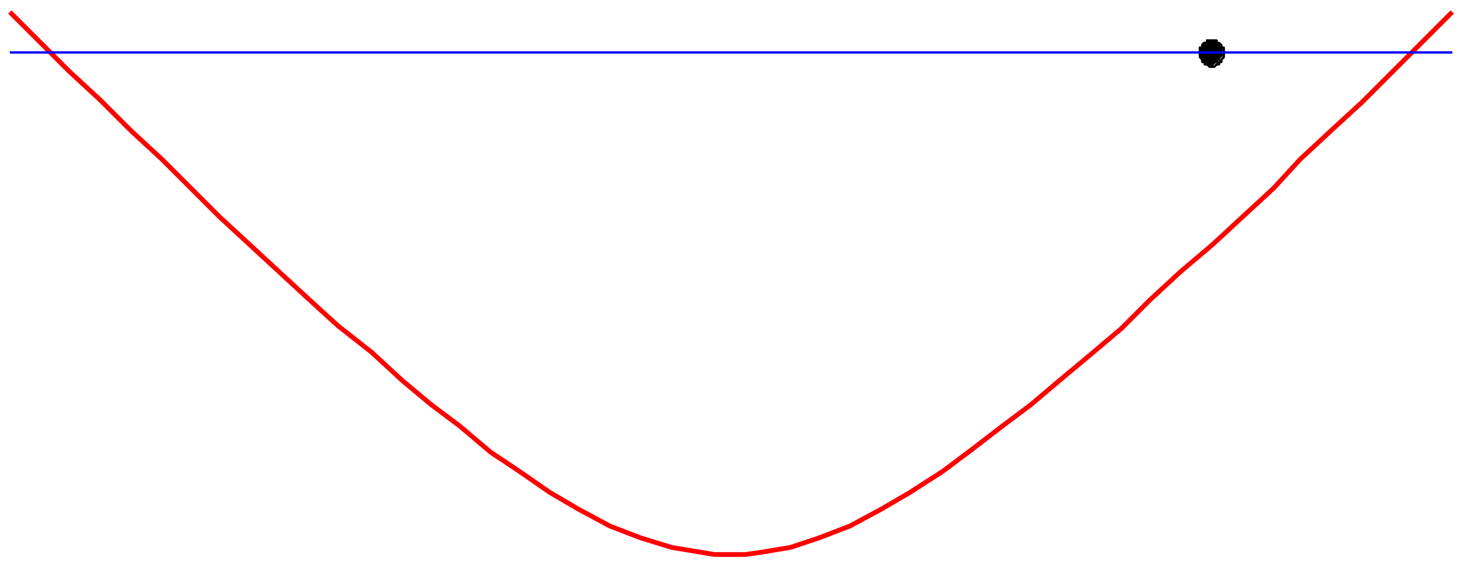

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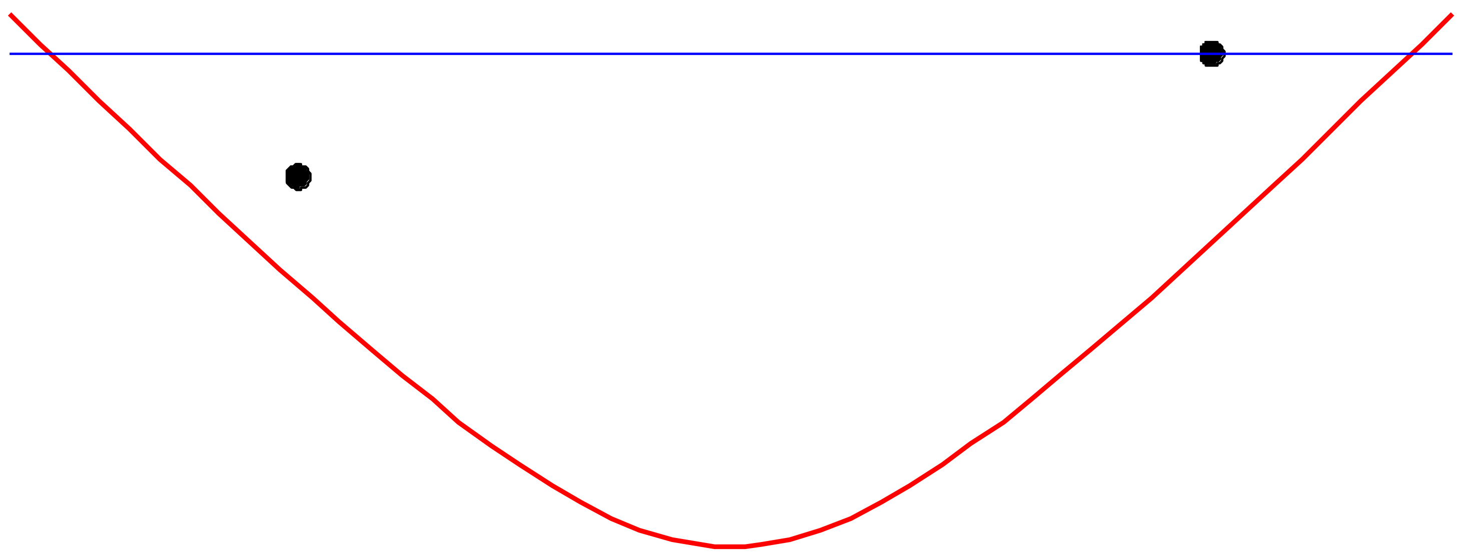

Because this tactic garanties that $S_{n+1} \subset S_n$.
The zone where the thief is allowed to move decreases at each step.

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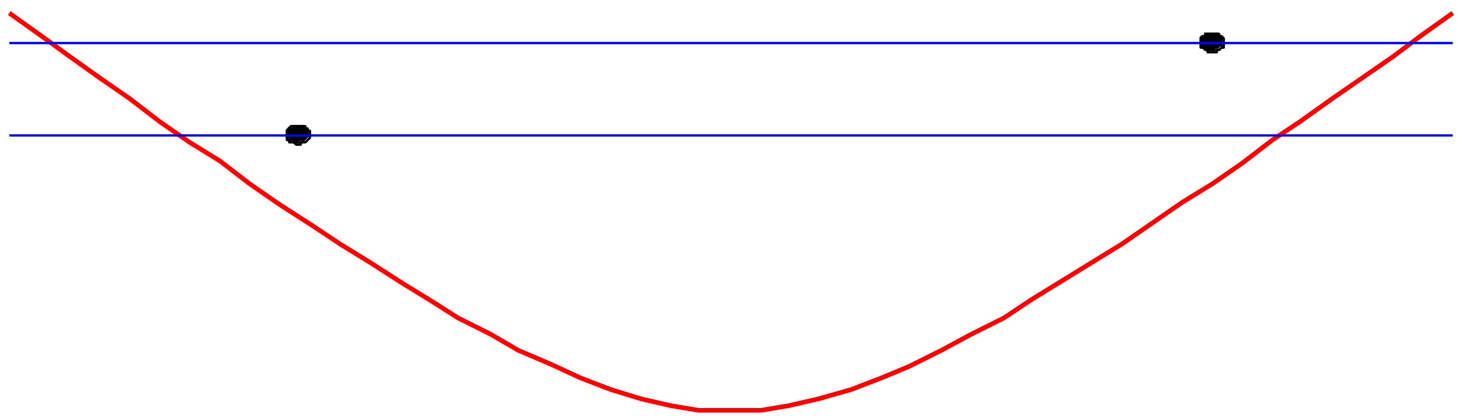

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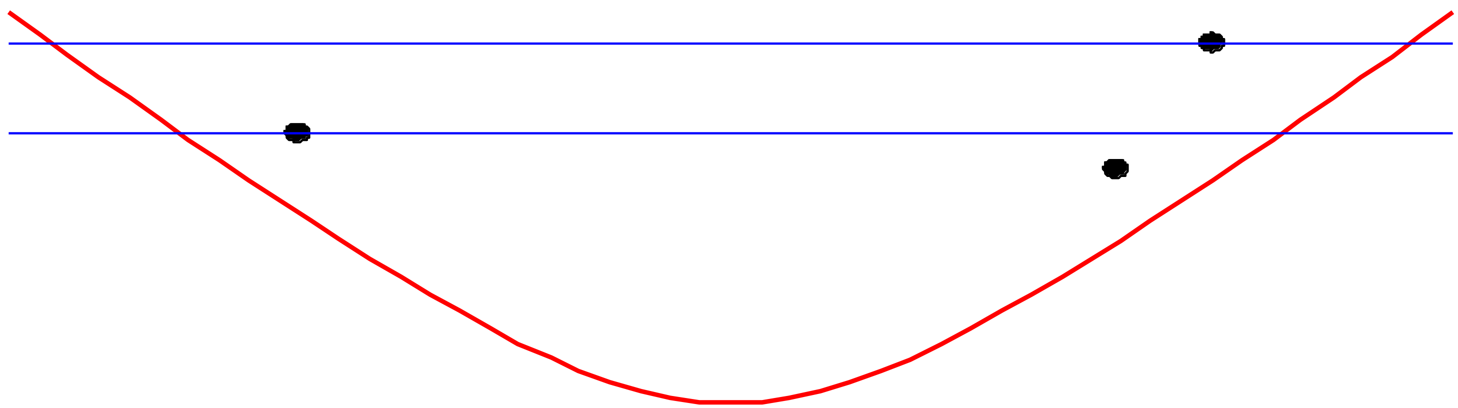

But the thief can move alternatively to the right and to the left (if $dim(X) \geq 2$).

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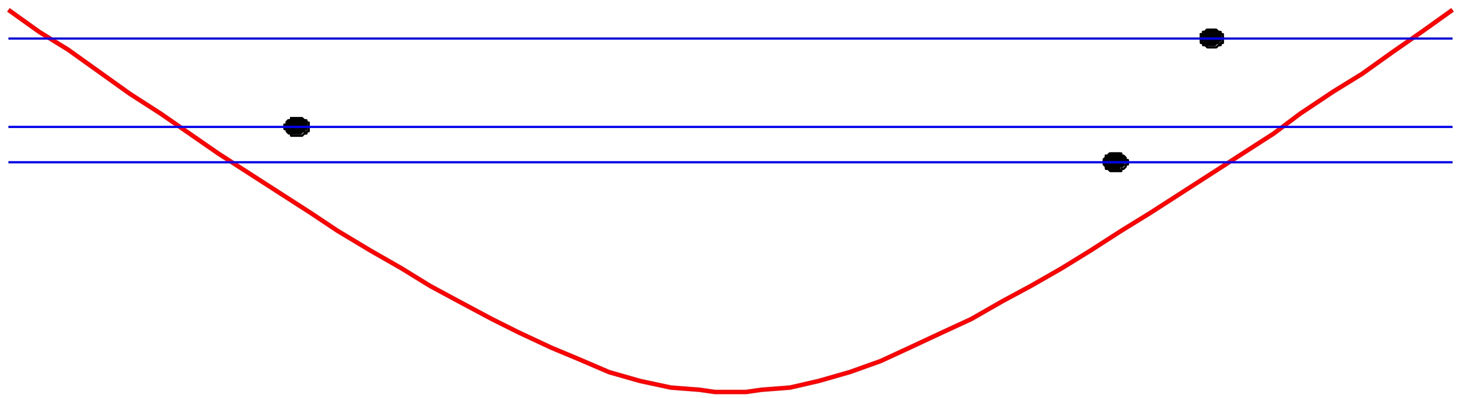

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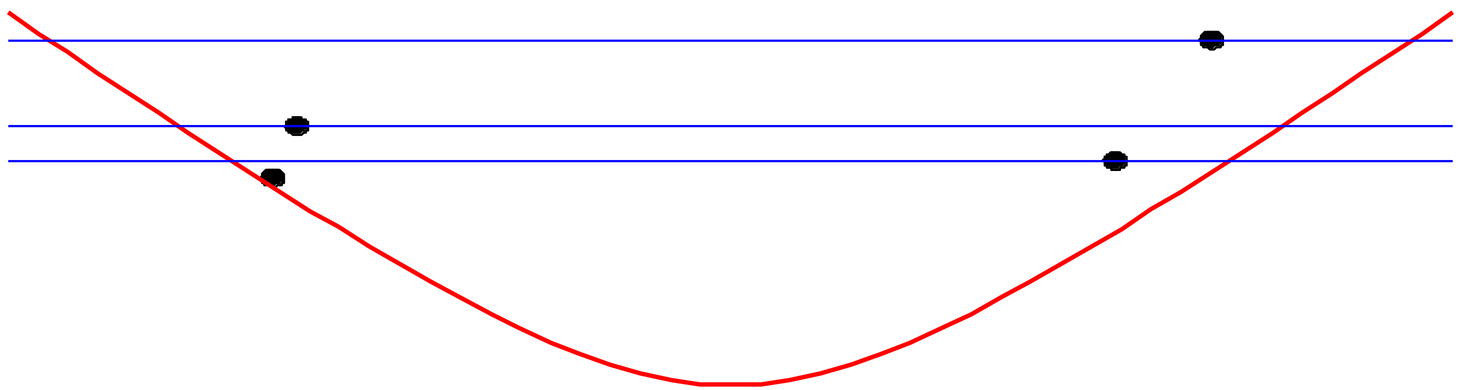

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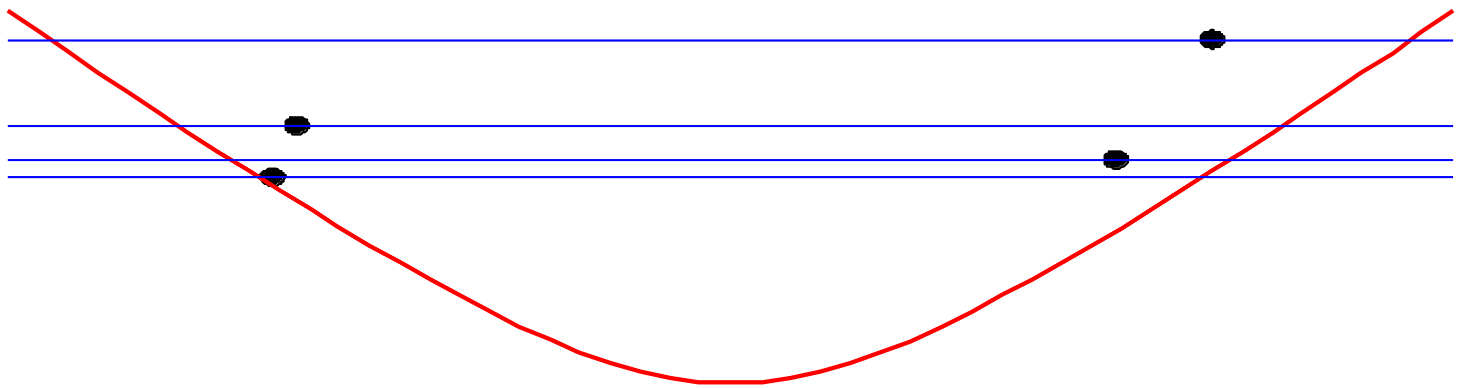

And the sequence (x_n) diverges if $\Big(f(x_n) - p\Big)$ does not go to 0.

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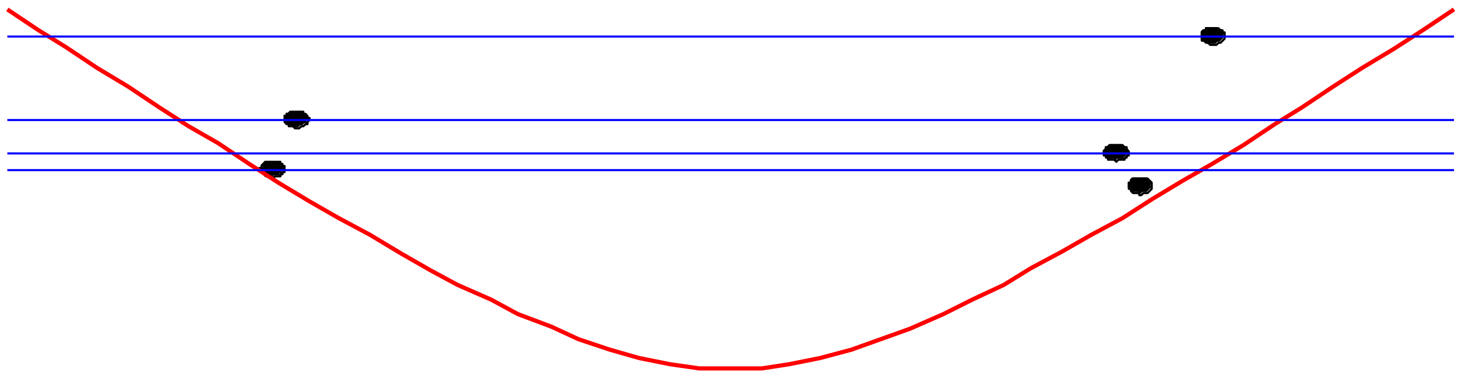
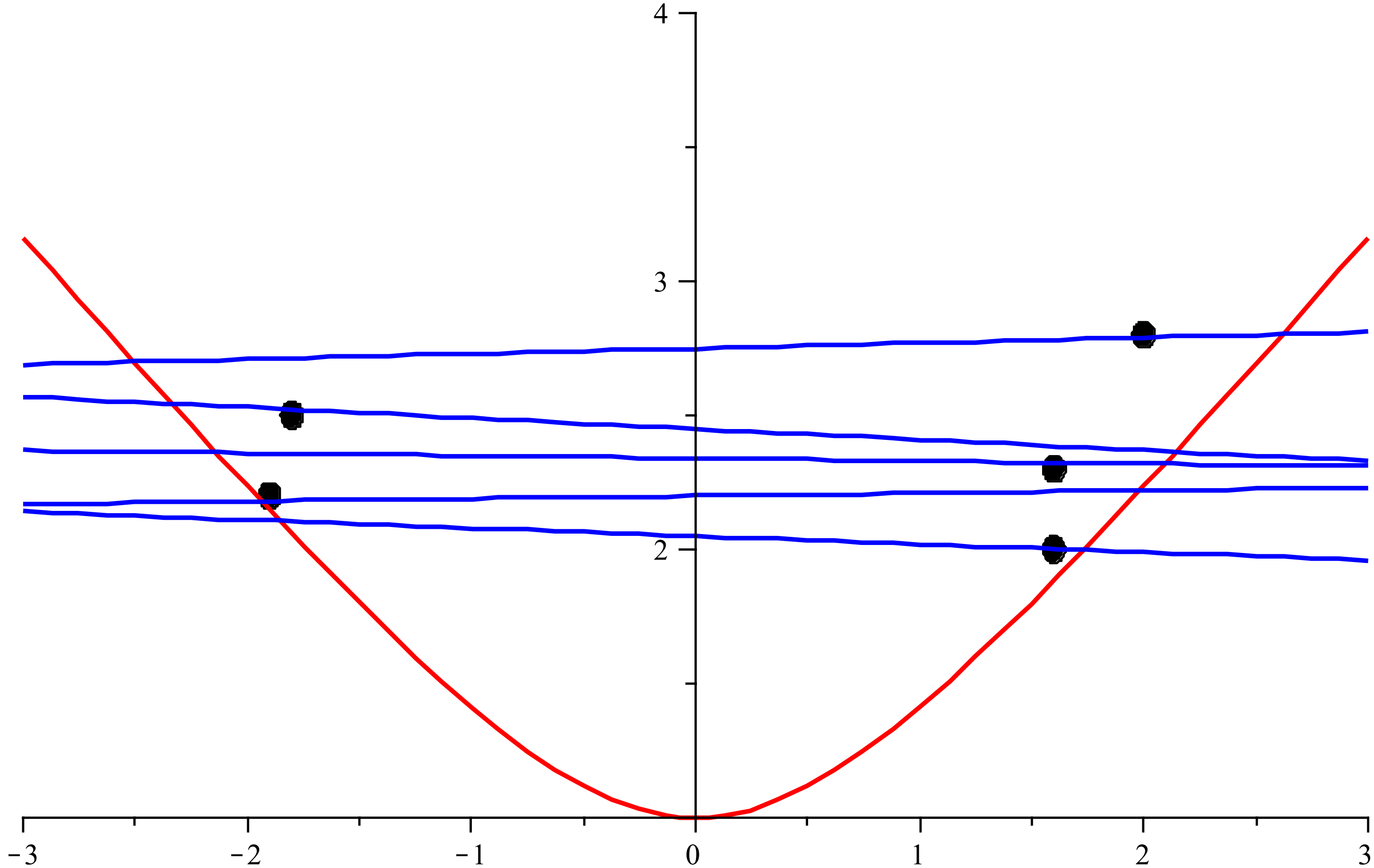


And the sequence (x_n) diverges if $\Big(f(x_n) - p\Big)$ does not go to 0.

Choosing $t(x_n)$ such that the sequence of corresponding slices S_n is decreasing is not a winning tactic for the policeman.



Related results.

Recall (Deville-Madiedo) :
If X has RNP, then for each $f \in S_{X^*}$ and each $\varepsilon > 0$, there exists $t : X \to S_{X^*} \cap B(f, \varepsilon)$ such that for all (x_n) in X, if $\Big(f(x_n) - \varepsilon \|x_n\|\Big)$ is bounded below and if $\langle\, t(x_n), x_{n+1} - x_n \,\rangle \leq 0$ for all n, then (x_n) converges.

Prochazka : If X is a Banach space with RNP and if C is a closed convex bounded subset of X, there exists $t_C : C \to S_{X^*}$ such that for any sequence $(x_n) \subset C$, if $\langle\, t_C(x_n), x_{n+1} - x_n \,\rangle \leq 0$ for all n, then (x_n) converges.

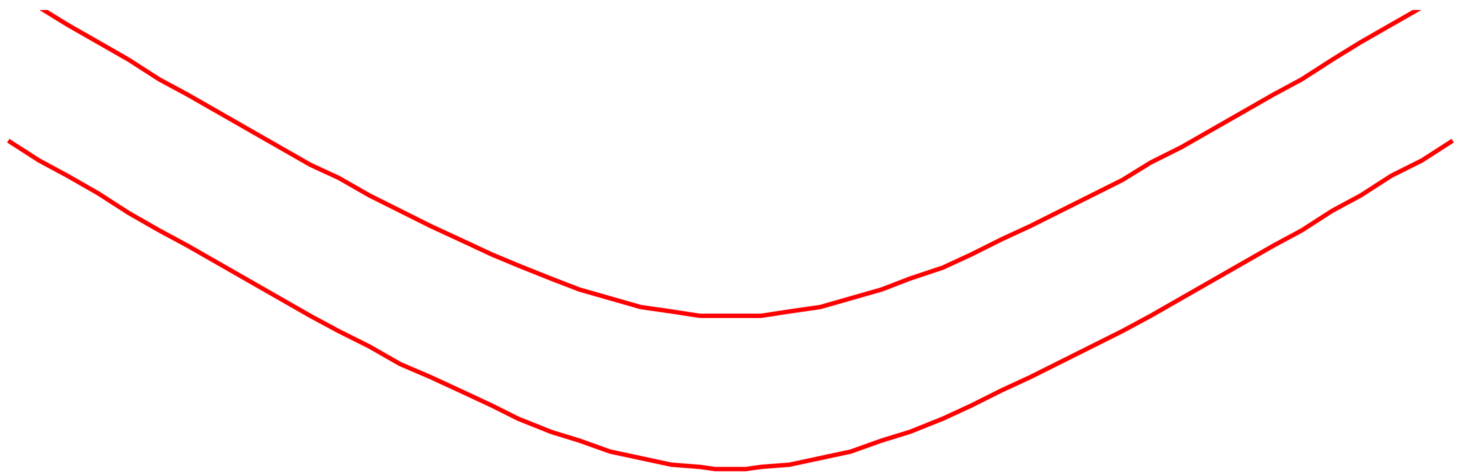
Zeleny : If $dim(X) < +\infty$, then t can be chosen continuous.

First results : Maly-Zeleny, Deville-Matheron.

Proof : η-tactics.

Fix $p \in \mathbb{Z}$ and $\eta > 0$, and let $\Lambda_p = \{x \in X : f(x) \geq \varepsilon\|x\| + p\}$.

We construct t on $\Lambda_p \backslash \Lambda_{p+1}$ such that whenever $(x_n) \subset \Lambda_p \backslash \Lambda_{p+1}$,
$\langle t(x_n), x_{n+1} - x_n \rangle \leq 0$ for all $n \quad \Rightarrow \quad (x_n)$ is η-Cauchy.



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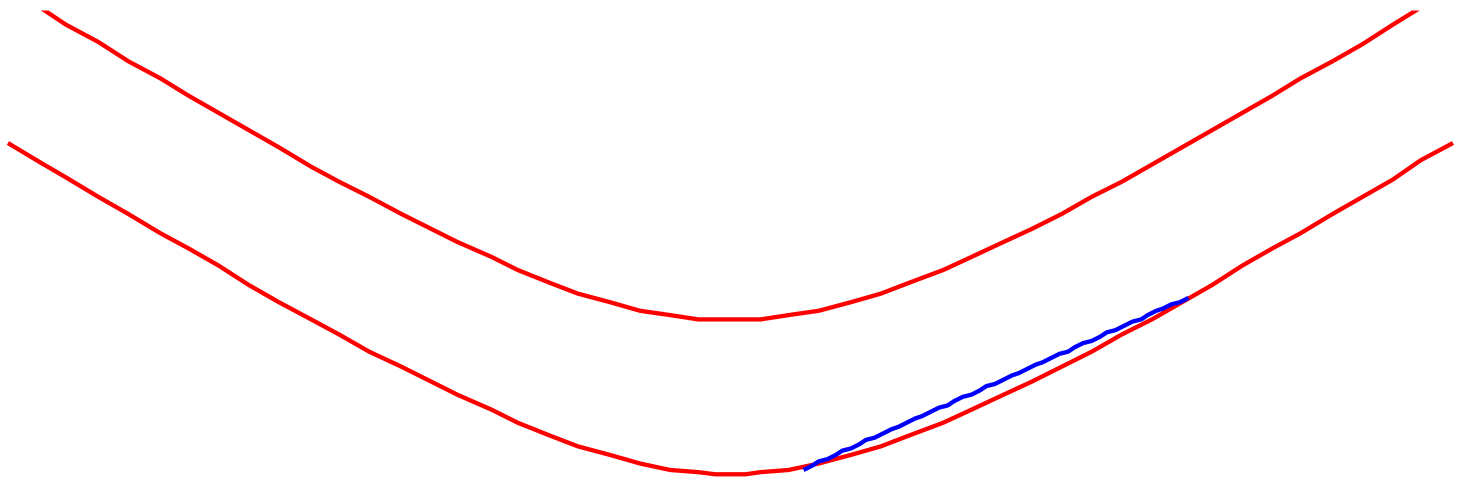

Let $f_1 \in B(f, \varepsilon)$ and $c_1 \in \mathbb{R}$ so that $C_1 = \Lambda_p \cap \{f_1 < c_1\} \neq \emptyset$, $diam(C_1) < \eta$, and $\Lambda_{p+1} \cap C_1 = \emptyset$. If $x \in C_1$, $t(x) = f_1$.

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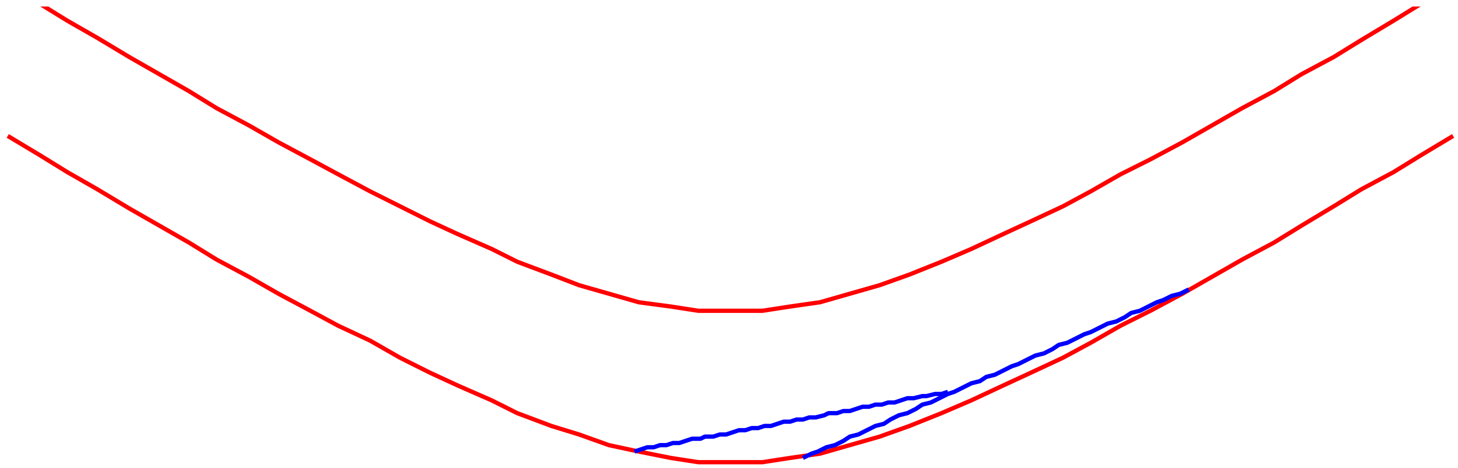

Let $f_2 \in B(f, \varepsilon)$ and $c_2 \in \mathbb{R}$ such that $C_2 = \left(\Lambda_p \backslash C_1\right) \cap \{f_2 < c_2\} \neq \emptyset$, $diam(C_2) < \eta$, and $\Lambda_{p+1} \cap C_2 = \emptyset$. If $x \in C_2$, $t(x) = f_2$.

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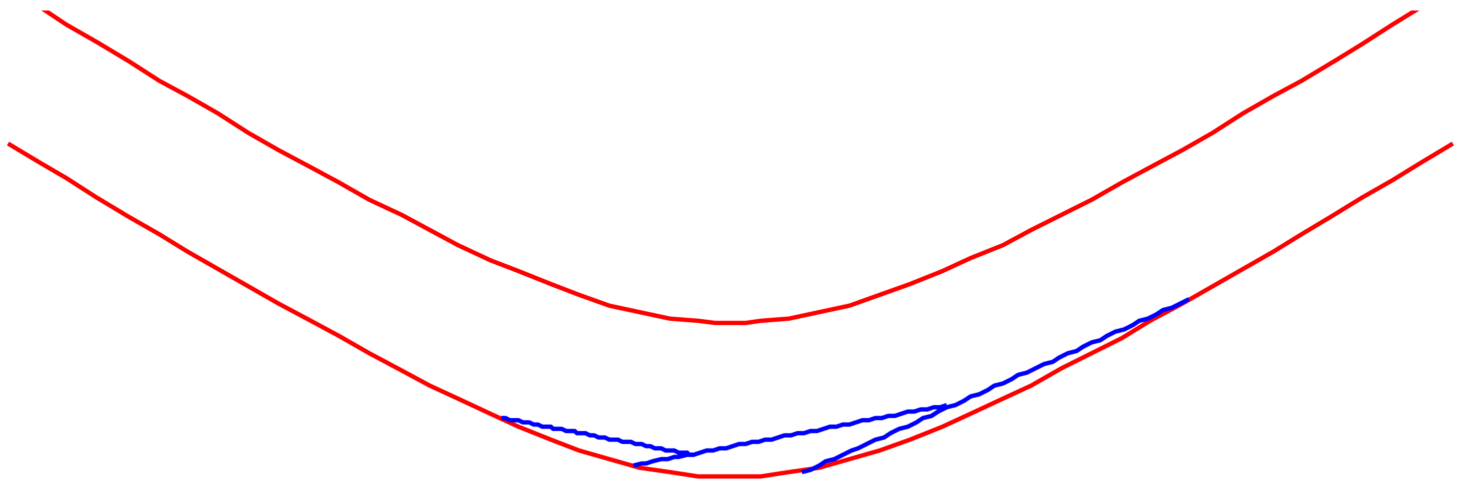
Fix $p \in \mathbb{Z}$ and $\eta > 0$, and let $\Lambda_p = \{x \in X : f(x) \geq \varepsilon\|x\| + p\}$.

We construct t on $\Lambda_p \backslash \Lambda_{p+1}$ such that whenever $(x_n) \subset \Lambda_p \backslash \Lambda_{p+1}$, $\langle t(x_n), x_{n+1} - x_n \rangle \leq 0$ for all $n \quad \Rightarrow \quad (x_n)$ is η-Cauchy.

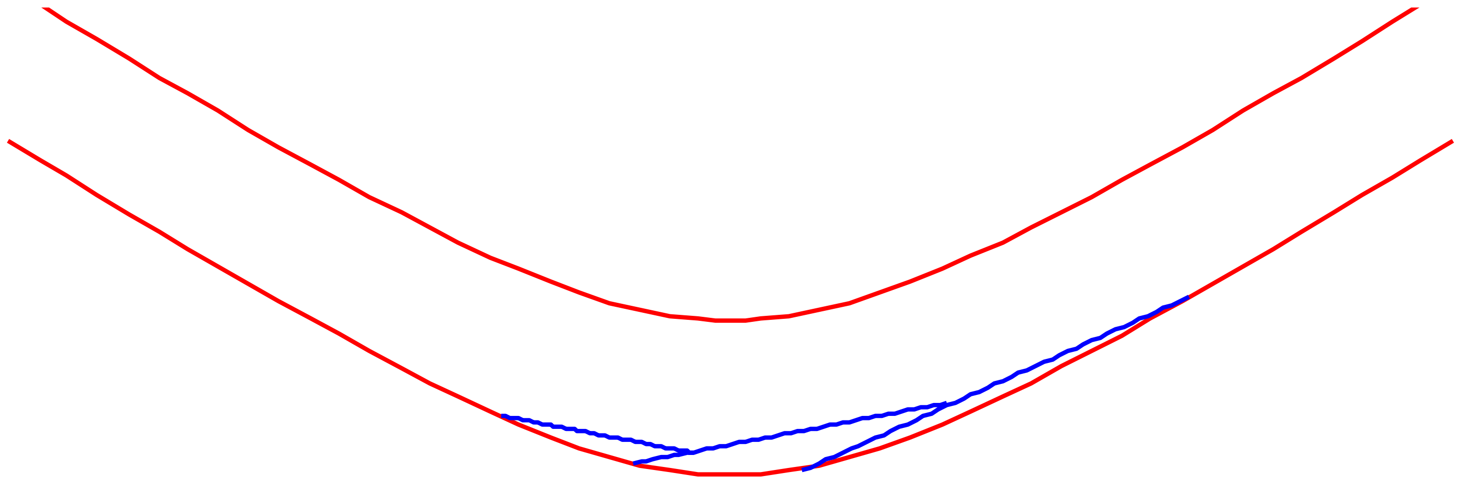


By transfinite induction, let $f_\alpha \in B(f, \varepsilon)$, $c_\alpha \in \mathbb{R}$ s. t. the associated convex sets C_α form a partition of $\Lambda_p \backslash \Lambda_{p+1}$. If $x \in C_\alpha$, $t(x) = f_\alpha$.

Proof : η-tactics.

If $x \in C_\alpha$, $t(x) = f_\alpha$.

If $(x_n) \subset \Lambda_p \backslash \Lambda_{p+1}$ satisfies $\langle\, t(x_n), x_{n+1} - x_n \,\rangle \leq 0$ for all n,
$\exists\, \alpha_n$ such that $x_n \in C_{\alpha_n}$.



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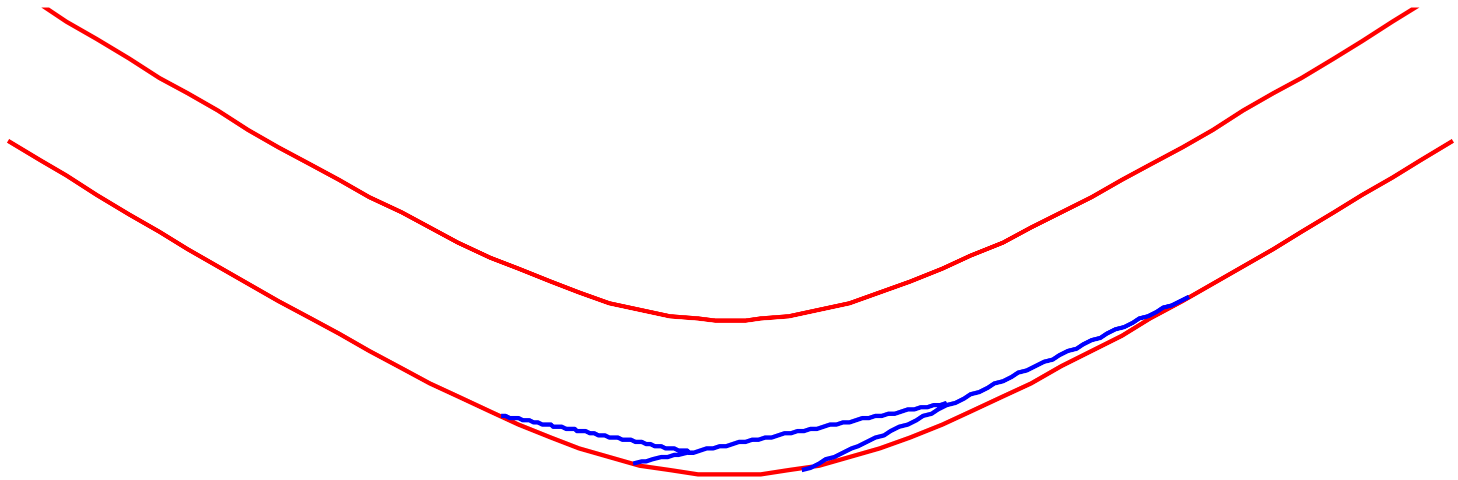

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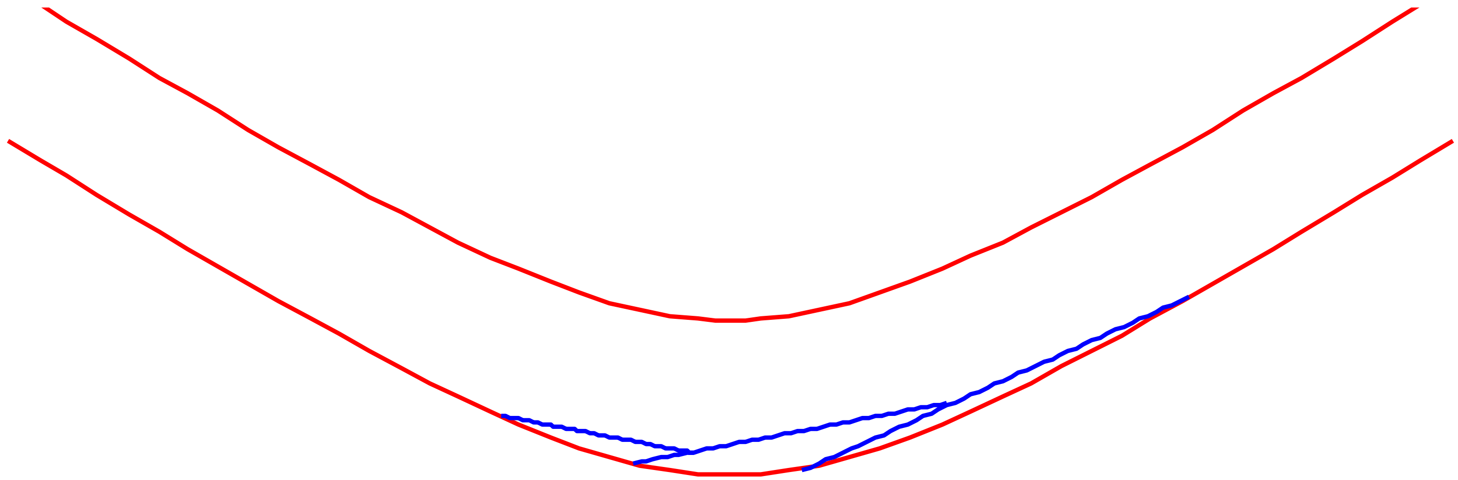
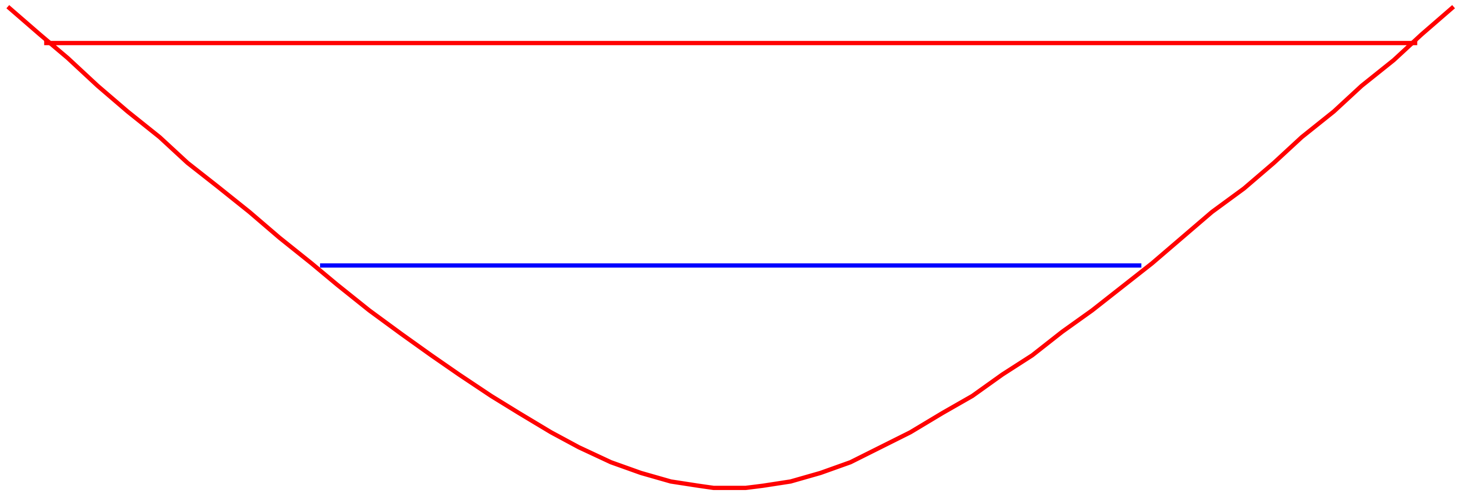


There exists n_0 such that for all $n \geq n_0$, $\alpha_n = \alpha_{n_0}$.
So $x_n \in C_{\alpha_{n_0}}$ for all $n \geq n_0$ and $diam(C_{n_0}) < \eta$: (x_n) is η-Cauchy.

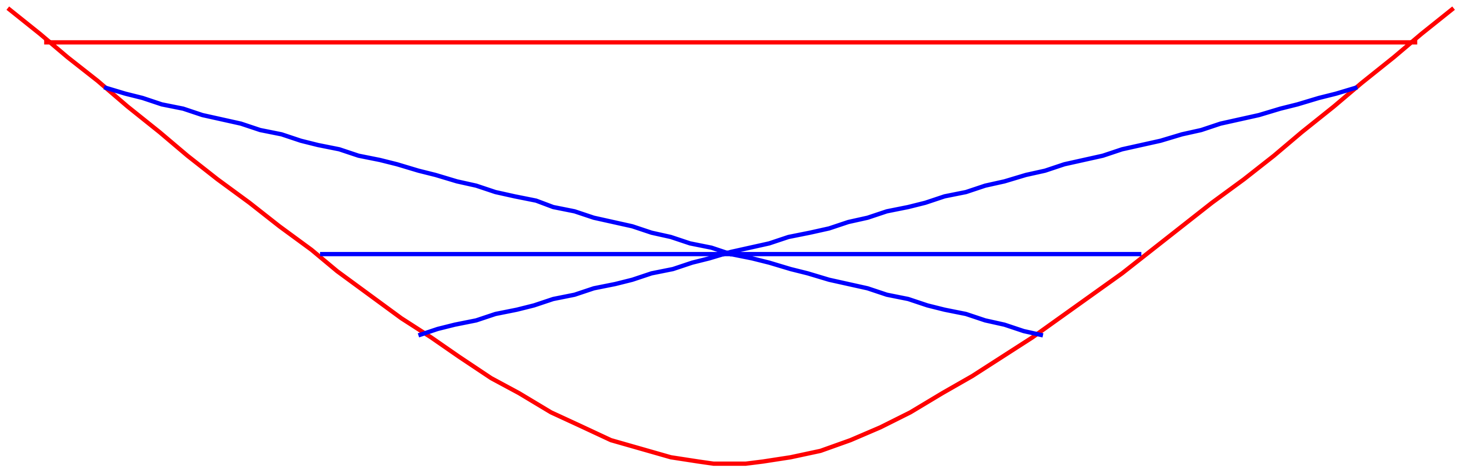
Proof : multi-η-tactics.



Representation of C_α, $\{f_\alpha = c_\alpha\}$ in red.

If $x \in C_\alpha$, $t(x) = f_\alpha$, t is an η-tactic.

Proof : multi-η-tactics.



Define $T(x) = \overline{B}\Big(f_\alpha, \delta(x)\Big)$ for $x \in C_\alpha$.

Any selection t of T is a η-tactic if $\delta(x) > 0$ is small enough.

Proof : Construction of t.

We construct multi-tactics T_k on $\Lambda_p \backslash \Lambda_{p+1}$ so that :

- $\forall x \in \Lambda_p \backslash \Lambda_{p+1}$ and $\forall k$, $T_{k+1}(x) \subset T_k(x) = \overline{B}(f_{k,x}, \delta_k(x)) \cap S_{X^*}$,
- If t is a selection of T_k, t is a η_k-tactic, where $(\eta_k) \downarrow 0$.
- $diam(T_k(x)) \to 0$.

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Assume that $\Big(f(x_n) - \varepsilon \|x_n\| \Big)$ is bounded below
and that $\langle\, t(x_n), x_{n+1} - x_n \,\rangle \leq 0$ for all n.

If $x_n \in \Lambda_{p_n} \backslash \Lambda_{p_n+1}$, $p_{n+1} \leq p_n$ and (p_n) is bounded below :
$p_n = p_{n_0}$ for all $n \geq n_0$.

The sequence $(x_n)_{n \geq n_0}$ is η_k Cauchy for all k, hence converges.

Application to differentiability.

Let Ω be open in $\mathbb{R}^d$ or in a Riemannian variety (M, g) of dimension $d \geq 2$. Let $F : \Omega \times \mathbb{R}^d$ (or TM) $\to \mathbb{R}$ continuous.

Definition. $u : \Omega \to \mathbb{R}$ is an almost classical solution of $F(x, Du(x)) = 0$ if u is diff. at **each** point of Ω,and if :
1) $F(x, Du(x)) = 0$ a. e.
2) $F(x, Du(x)) \leq 0$ for all $x \in \Omega$.

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Theorem. (with J. Jaramillo) Suppose that
A) $\exists\, u_0 : \Omega \to \mathbb{R}$ $\mathcal{C}^1$, so that $F(x, Du_0(x)) \leq 0$ for all $x \in \Omega$.
B) $\exists\, \rho : \Omega \to (0, +\infty)$ locally bounded, such that
$\{v;\, F(x, v) \leq 0\} \subset \overline{B}(0, \rho(x))$ for all $x \in \Omega$.
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Example. If $d \geq 2$, $\exists u : S^d \to \mathbb{R}$ differentiable at each point, so that $\|Du(x)\| = 1$ a. e.

Mountain smooth and steep almost everywhere !