Lineability problems motivated by a classical characterization of continuity

Gustavo Adolfo Muñoz Fernández

GENERICITY & SMALL SETS IN ANALISYS

Joint work with

J. L. Gámez, D. Pellegrino, V. M. Sánchez and J. B. Seoane

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A characterization of continuity

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is continuous then

1. f transforms connected sets into connected sets.
2. f transforms compact sets into compact sets.

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is a function such that

1. f transforms connected sets into connected sets,
2. f transforms compact sets into compact sets, and

then f is necessarily continuous.

A characterization of continuity

Motivation for the generalization

- A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in $\mathbb{R}$ if and only if

$$f^{-1}(U) \text{ is open for all open set } U \subset \mathbb{R}.$$

- Is the previous definition equivalent to

$$f(U) \text{ is open for all open set } U \subset \mathbb{R}?$$

- Obviously, the answer is NO, but:
- Is there a family $\mathcal{F}$ of subsets of $\mathbb{R}$ such that

$$f \text{ is continuous if and only if } f(U) \in \mathcal{F} \text{ for all } U \in \mathcal{F}.$$

- The answer again is no, but the result is highly nontrivial.

A characterization of continuity

Theorem (Velleman (1997))

There are not families $\mathcal{F}$ and $\mathcal{G}$ of subsets of $\mathbb{R}$ such that

$$f : \mathbb{R} \to \mathbb{R} \text{ is continuous if and only if } f(U) \in \mathcal{G} \text{ for all } U \in \mathcal{F}.$$

Theorem (Velleman (1997), Hamlett (1975), White (1968))

There are two families $\mathcal{F}$ and $\mathcal{G}$ of subsets of $\mathbb{R}$ such that $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

1. $f(U) \in \mathcal{F}$ for all $U \in \mathcal{F}$, and
2. $f(V) \in \mathcal{G}$ for all $V \in \mathcal{G}$.

A characterization of continuity

The characterization again

A plausible choice for $\mathcal{F}$ and $\mathcal{G}$ in the previous theorem is the following:

1. $\mathcal{F}$ is the family of all connected subsets of $\mathbb{R}$ (the intervals), and
2. $\mathcal{G}$ is the family of all compact subsets of $\mathbb{R}$.

A generalization of the characterization

1. The same result holds for functions $f : X \to Y$ where X is first countable and locally connected and Y is regular.
2. However the result is not true for functions between metric spaces in general.

Derivatives as connected functions

Taken from a test for Analysis I students

Let f be differentiable in $I = (a, b)$ and $c \in I$. Then:

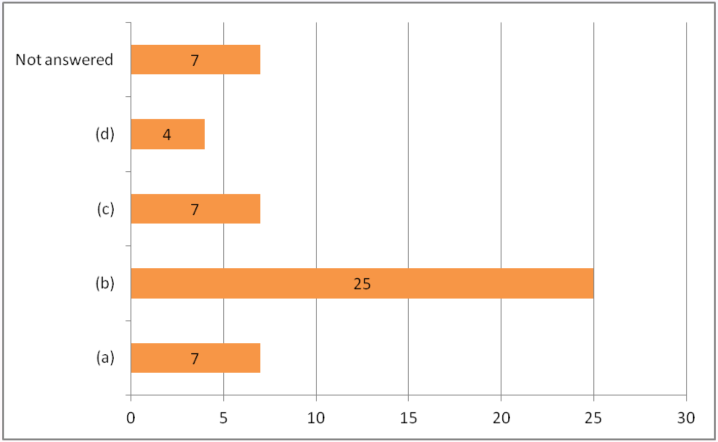
(a) f is uniformly continuous I.

(b) $\lim_{x \to c} f'(x) = f'(c)$.

(c) $f'(I)$ is an interval.

(d) None of the previous answers are correct.

Outcome of the test question



Derivatives as connected functions

Theorem (Darboux)

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is a Darboux functions, i.e., f' transforms intervals into intervals.

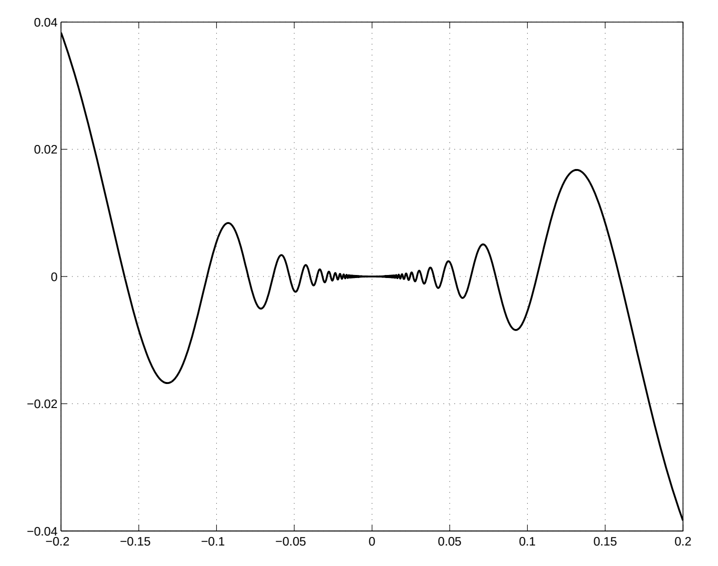
Derivatives are not necessarily continuous

The derivative of

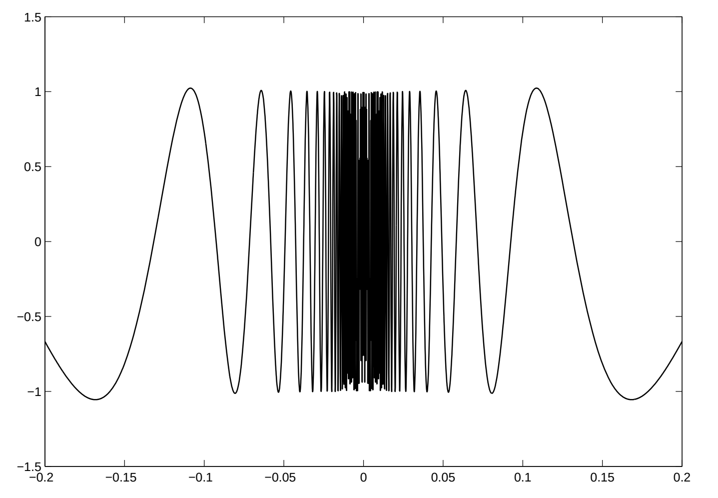
$$G(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0.

The graph of $G(x)$



The graph of $G'(x)$



Derivative with uncountably many discontinuities

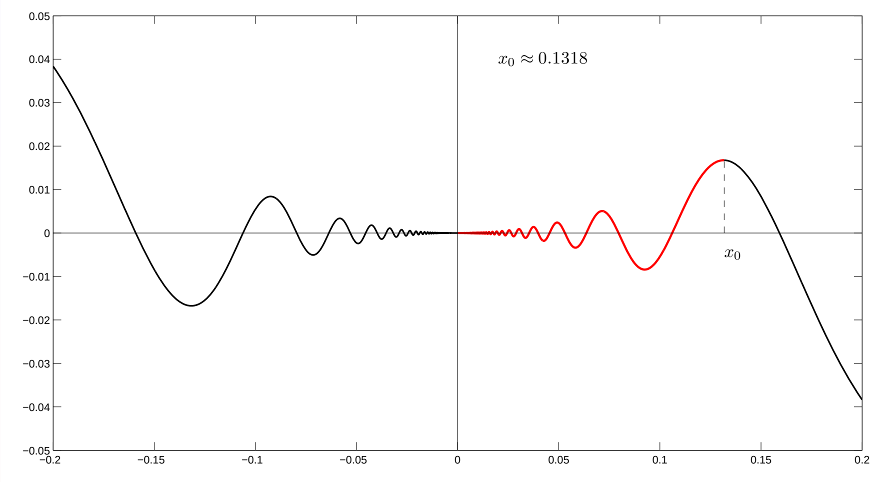
Volterra construction

1. Choose $x_0 > 0$ so that $G'(x_0) = 0$.
2. Define $G_0 : (0, 2x_0) \to \mathbb{R}$ as follows:

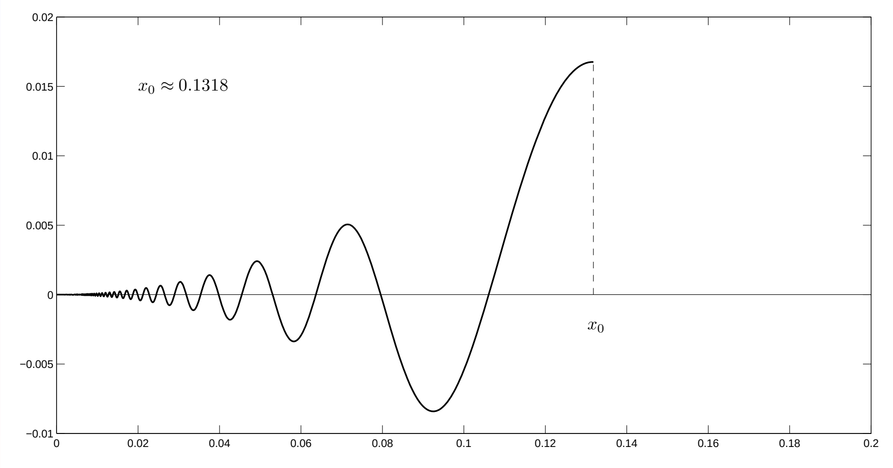
$$G_0(x) = \begin{cases} G(x) & \text{if } x \in (0, x_0], \\ G(2x_0 - x) & \text{if } x \in [x_0, 2x_0). \end{cases}$$

3. Using translations and homothetic transformations of G_0, F coincides with a copy of G_0 in every interval (a, b) of $[0, 1] \setminus C$, where C is the Cantor set.
4. We put $F(x) = 0$ for all $x \in C$.
5. F is differentiable in $[0, 1]$ but F' is not continuous in C.

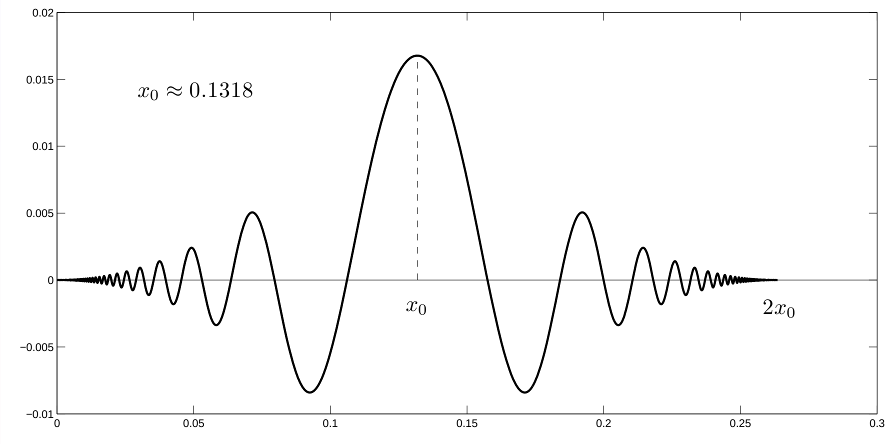
Visual Volterra construction



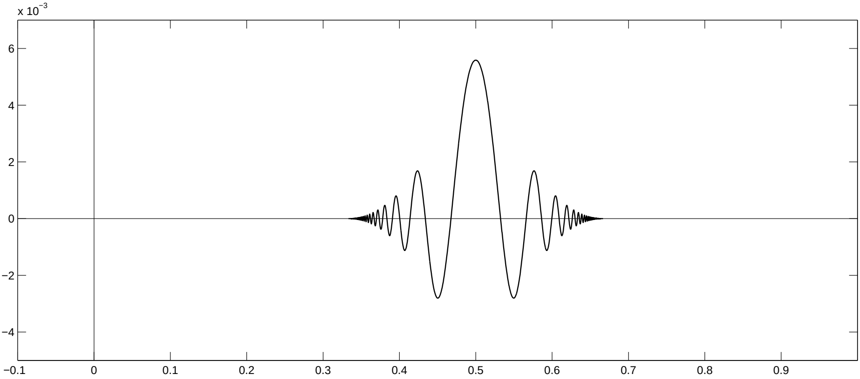
Visual Volterra construction



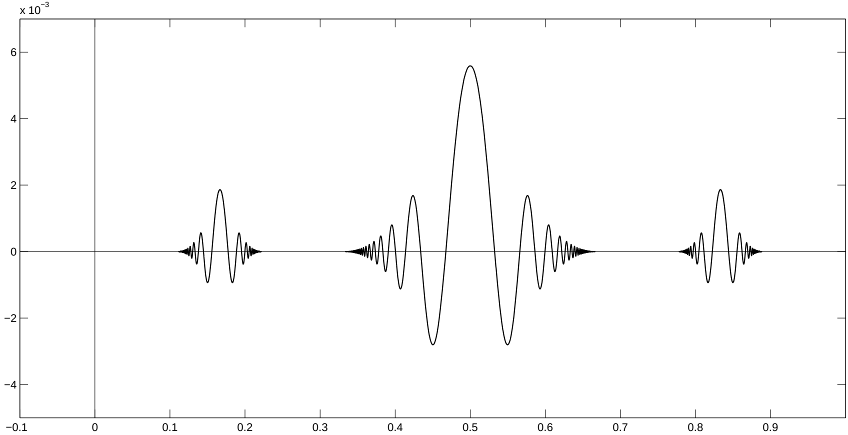
Visual Volterra construction: G_0



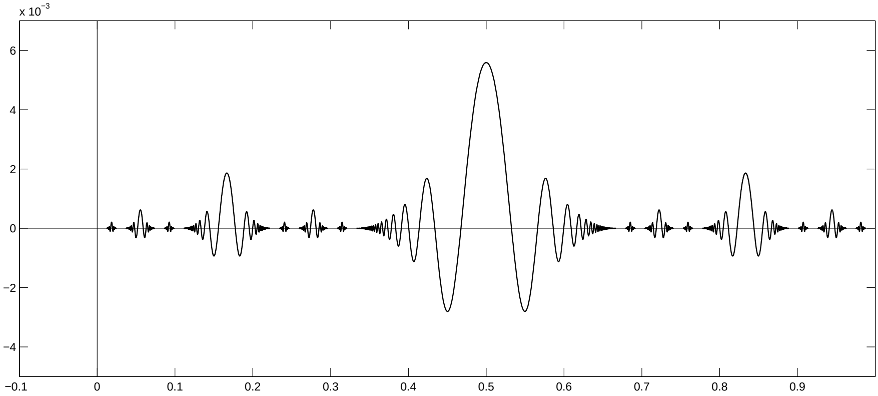
Visual Volterra construction: Replication of G_0



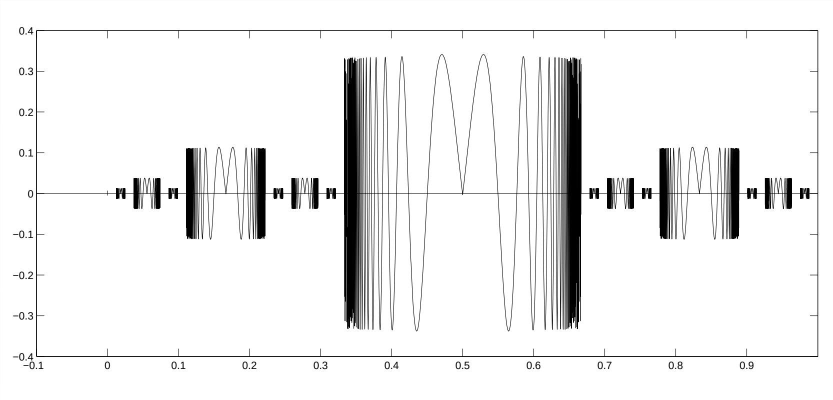
Visual Volterra construction: Replication of G_0



Function whose derivative has uncountably many discontinuities



Derivative with uncountably many discontinuities



More about Volterra constructions

Remark

1. The Volterra construction goes back to 1881.
2. If $\epsilon \in [0,1)$, there is a Volterra type construction F_ϵ over any Cantor type set in $[0,1]$ of measure ϵ.
3. The function F' is not Riemann integrable if we choose a Cantor set of positive measure.
4. If $E \subset [0,1]$ is nowhere dense, then there is a Volterra type construction in $[0,1]$ whose derivative is discontinuous in E.

Theorem

If $A \subset [0,1]$ is a dense G_δ set then there exists $f : [0,1] \to \mathbb{R}$ such that f' is continuous in A and discontinuous in $[0,1] \setminus A$.

More about Volterra constructions

Sketch of the proof

1. $[0,1] \setminus A = \cup_{n=1}^{\infty} E_n$ with E_n nowhere dense.
2. Consider F_n a Volterra construction for E_n with oscillation 1 at all points of E_n.
3. $f = \sum_{n=1}^{\infty} \frac{F_n}{2^n}$ is the desired function.

Theorem

1. *The set of continuity points of a function in* $[0,1]$ *is a* G_δ *set.*
2. *The set of continuity points of a derivative in* $[0,1]$ *is a dense* G_δ *set.*

Corollary

A set $A \subset [0,1]$ *is the set of continuity points of a derivative defined on* $[0,1]$ *if and only if* A *is a dense* G_δ *set.*

Pompeiu derivatives

Definition

A Pompeiu derivative in $[0, 1]$ is a derivative with a dense set of zeros.

Remark

1. If $f : [0, 1] \to \mathbb{R}$ is a Pompeiu derivative, $x_0 \in [0, 1]$ and $f'(x_0) \neq 0$, then f' is not continuous at x_0.
2. In principle, Pompeiu derivatives do not necessarily have many discontinuities.
3. Pompeiu managed to construct a non null Pompeiu derivative in 1907.

Theorem

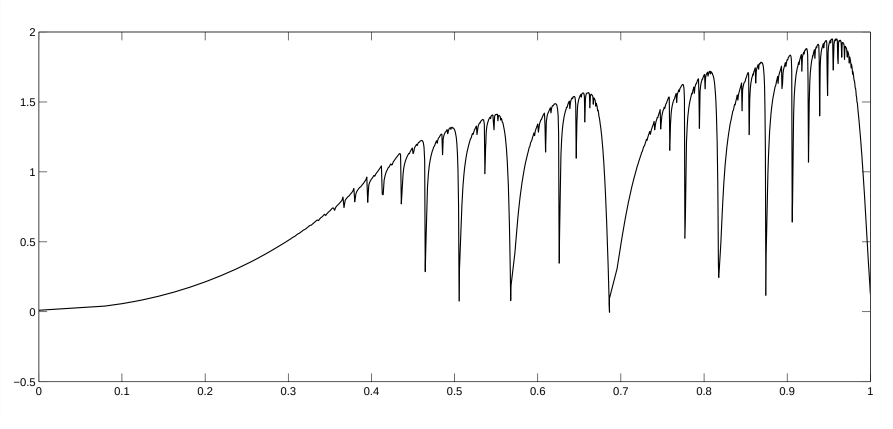
The bounded Pompeiu derivatives on $[0, 1]$ *form a Banach space with he uniform norm.*

Pompeiu's original example (1907)

Pompeiu derivative with many discontinuities

1. Let $\{r_n : n \in \mathbb{N}\}$ an ordering of $[0,1] \cap \mathbb{Q}$.
2. Let $\sum_{n=1}^{\infty} \alpha_n$ be a convergent series of positive real numbers.
3. Define $g(x) = \sum_{n=1}^{\infty} \alpha_n (x - r_n)^{\frac{1}{3}}$ in $[0,1]$.
4. g is continuous and strictly increasing.
5. $g'(x) = \sum_{n=1}^{\infty} \frac{1}{3} \alpha_n (x - r_n)^{-\frac{2}{3}} > 0$ whenever the sum is finite.
6. $g'(x) = \infty$ whenever $\sum_{n=1}^{\infty} \frac{1}{3} \alpha_n (x - r_n)^{-\frac{2}{3}} = \infty$.
7. In particular $g'(r_n) = \infty$.
8. Normalize g by defining $G(x) = \frac{g(x) - g(0)}{g(1) - g(0)}$ in $[0,1]$.
9. Define $F = G^{-1}$ on $[0,1]$.
10. F is everywhere differentiable in $[0,1]$ and $F'(r_n) = 0$.
11. Hence F' is a Pompeiu derivative.

Graph of the Pompeiu's original example



Derivatives that are discontinuous almost everywhere

Definition (Aron, Gurariy, and Seoane (2004))

A subset V of a linear space E is λ-lineable if $V \cup \{0\}$ contains a linear space of dimension λ.

Theorem (Gámez, Muñoz, Sánchez, and Seoane (2010))

The set of differentiable functions on $\mathbb{R}$ whose derivatives are discontinuous almost everywhere is $\mathfrak{c}$-lineable.

Corollary

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous almost everywhere is $\mathfrak{c}$-lineable.

Derivatives that are discontinuous almost everywhere

Definition

Let $E \subset \mathbb{R}$. We say that $x \in \mathbb{R}$ is a point of density of E if

$$\liminf_{\varepsilon \to 0^+} \frac{m(E \cap (x-\varepsilon, x+\varepsilon))}{2\varepsilon} = 1,$$

where m stands for the Lebesgue measure on $\mathbb{R}$. We denote

$$\operatorname{dens} E = \{\, x \in \mathbb{R} \,:\, x \text{ is a density point of } E \,\}.$$

Definition

We say that $f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at $x_0 \in \mathbb{R}$, if there exists $E \subset \mathbb{R}$ such that $x_0 \in \operatorname{dens} E$ and

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = f(x_0).$$

Derivatives that are discontinuous almost everywhere

Lemma (Zahorki)

There exists an approximately continuous mapping $f_0 \colon \mathbb{R} \to [0,1]$ *satisfying the following properties:*

1. Z_{f_0} *is a* G_δ*, dense set with Lebesgue measure zero.*
2. f_0 *is discontinuous at every* $x \in \mathbb{R} \setminus Z_{f_0}$.

Theorem (Gámez, Muñoz, Sánchez, and Seoane (2010))

The set of bounded approximately continuous mappings defined on $\mathbb{R}$ *that are discontinuous almost everywhere is* $\mathfrak{c}$*-lineable.*

Sketch of the proof

The set $\{f_0(x)e^{-\alpha|x|} : \alpha \in (0,\infty)\}$ is a basis of cardinality $\mathfrak{c}$.

Derivatives that are discontinuous almost everywhere

Theorem

All bounded approximately continuous functions on $\mathbb{R}$ are derivatives.

Corollary (Gámez, Muñoz, Sánchez, and Seoane (2010))

The set of bounded Pompeiu derivatives on $\mathbb{R}$ that are discontinuous almost everywhere is $\mathfrak{c}$-lineable.

Everywhere surjective functions

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is everywhere surjective if $f(I) = \mathbb{R}$ for all nontrivial interval I.

Theorem (Aron, Gurariy, and Seoane (2004))

The set of everywhere surjective functions on $\mathbb{R}$ is $2^{\mathfrak{c}}$-lineable.

Corollary

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that transform connected sets into connected sets and are discontinuous everywhere is $2^{\mathfrak{c}}$-lineable.

Functions that transform compact sets into compact sets

Theorem (Gámez, Muñoz, and Seoane (2011))

The set of functions $f : \mathbb{R} \to \mathbb{R}$ that have finite range (and hence transform any set into a compact set) and are everywhere discontinuous is $2^{\mathfrak{c}}$-lineable.

Sketch of proof

- Let H be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$.
- Let $\varphi : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ a $\mathbb{Q}$-linear isomorphism.
- For all $A \subset H$ we define $f_A(x) := \chi_{([A]\setminus\{0\})^{\mathbb{N}}}(\varphi(x))$, for all $x \in \mathbb{R}$.
- Choose $h_0 \in H$ and consider $F = \{f_A : \varnothing \neq A \in \mathcal{P}(H),\ h_0 \notin A\}$. Then F is linearly independent and its cardinality is $2^{\mathfrak{c}}$.

Polynomials in finite variables

- Multiindex: $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$.
- Trace of a multiindex: $|\alpha| = \alpha_1 + \cdots + \alpha_m$.
- Monomial $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{K}^m$.

Definition (Polynomials on $\mathbb{K}^m$)

A polynomial of degree at most n in $\mathbb{K}^m$ is given by

$$P(\mathbf{x}) = \sum_{|\alpha| \leq n} \mathbf{x}^\alpha.$$

A homogeneous polynomial of degree n in $\mathbb{K}^m$ is given by

$$P(\mathbf{x}) = \sum_{|\alpha| = n} \mathbf{x}^\alpha.$$

Polynomials on a normed space

Definition (Polynomials in infinitely many variables)

If E is a vector space (possible infinite dimensional), we say that $P : E \to \mathbb{K}$ is an n-homogeneous polynomial on E if there exists an n-linear form L on E such that for all $x \in E$

$$P(x) = L(x, \ldots, x).$$

A polynomial P of degree at most n on E is defined as

$$P = P_n + \cdots + P_1 + P_0,$$

where the P_k's are k-homogeneous and $P_0 \in \mathbb{K}$.

Polynomials on a normed space

Theorem

$P : E \to \mathbb{K}$ is a polynomial of degree at most n (n-homogeneous) if and only if for any choice $e_1, \dots e_m \in E$

$$\mathbb{K}^m \ni (x_1, \dots, x_m) \mapsto P(x_1 e_1 + \cdots + x_m e_m),$$

is a polynomial of degree at most n (n-homogeneous) in $\mathbb{K}^m$.

Theorem (Polarization Formula)

If P is an n-homogeneous polynomial on E then there exists a unique **symmetric** *n-linear form on E (the* **polar** *of P) such that $P(x) = L(x, \dots, x)$ for all $x \in E$. Moreover*

$$L(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} P(\epsilon_1 x_1 + \cdots + \epsilon_n x_n).$$

Polynomials on a normed space

Theorem

A polynomial P (resp. a multilinear form L) on a normed space E is continuous if and only if P (resp. L) is bounded on the unit ball of E.

We use the standard notations $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ and $\mathcal{L}^s(^nE)$ endowed with the sup norm over the unit ball of E.

Theorem (Martin, 1932)

If $P \in \mathcal{P}(^nE)$ and $L \in \mathcal{L}^s(^nE)$ is its polar then

$$\|L\| \leq \frac{n^n}{n!}\|P\|,$$

and the constant cannot generally be improved.

A characterization of continuity for polynomials

Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011))

If E is a normed space and P is a polynomial on E then P is continuous if and only it transforms compact sets into compact sets.

Sketch of the proof

1. Suppose $\lim_n x_n = 0$ but $\lim_n P(x_n) = a \neq 0$.
2. Two possibilities are plausible:
3. $P(x_n) \neq a$ for infinitely many n's:
    1. $\exists (y_n)$ a subsequence such that $P(y_n) \neq a$ for all n's.
    2. $C := \{y_n\} \cup \{0\}$ is compact but $P(C)$ is not.
4. Assume that $P(x_n) = a$ for all n.
    1. $\exists (y_n)$ with $P(y_n) \neq a \; \forall n$, $\lim_n P(y_n) = a$ and $\lim_n y_n = 0$.
    2. Again, $C := \{y_n\} \cup \{0\}$ is compact but $P(C)$ is not.

A characterization of continuity for polynomials

Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011))

If E is a normed space and $P \in \mathcal{P}(^n E)$ with $n = 1, 2$, then P is continuous if and only it is connected.

Sketch of the proof for 2-homogeneous polynomials

1. Suppose $\lim x_n = 0$ but $P(x_n) \uparrow \infty$ with $P(x_1) > 0$.
2. Consider the connected set $C := (\bigcup_{n=1}^{\infty} [x_n, x_{n+1}]) \cup \{0\}$.
3. $P([x_n, x_{n+1}]) \subset [P(x_n), \infty)$.
4. $P(C) = [P(x_1), \infty) \cup \{0\}$ which is not connected!!

Conjecture

A polynomial P on a normed space E is continuous if and only if it transforms connected sets into connected sets.

A characterization of continuity for multilinear forms

Corollary (Gámez, Muñoz, Pellegrino, and Seoane (2011))

An n-linear form L on a normed space E is continuous if and only if it transforms connected set in E^n into connected sets in $\mathbb{R}$.

Theorem (Gámez, Muñoz, Pellegrino, and Seoane (2011))

If $n \in \mathbb{N}$ and E is a normed space of infinite dimension λ, then the sets of the non-bounded n-linear forms, the non-bounded n-linear symmetric forms, the n-homogeneous polynomials and the polynomials of degree at most n on E are 2^λ-lineable.