Generic boundary behaviour for harmonic functions in the ball

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- If $f \in L^p(\mathbb{T})$, $p > 1$, $S_n f(x) = \sum_{k=-n}^{n} \hat{f}(k)e^{inx}$ is almost surely convergent but there are possible divergence points.
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• For a given $\beta$, what is the size of the set of points $x$ for which $|S_n f(x)| \gg n^\beta$ i.o.?
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If \( f \) is a generic function in \( L^p(\mathbb{T}) \),

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\text{for any } \beta \in [0, 1/p], \quad \dim_{\mathcal{H}}(E(\beta, f)) = 1 - \beta p.
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- $h(re^{ix}) = P_r * f(x)$ is harmonic in the unit disk.
- $r \to 1$ corresponds to the radial convergence in the disk.
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Fatou’s Lemma

Fatou (1906): If $f \in L^\infty(T)$, then $\mathcal{P} \mathcal{R} f(x) \to f(x)$ almost surely.

Generalizations (Hardy-Littlewood, Wiener, Bochner, ...)

Hunt and Wheeden (1970): If $h$ is a nonnegative harmonic function in a Lipschitz domain $U \subset \mathbb{R}^n$, then $h$ has a non-tangential limit at almost every point of the boundary $\partial U$. 
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Natural questions

Question
Let \( y \in S_d \) such that \( P[f](ry) \) diverges. How quick can be the divergence of \( P[f](ry) \) ?

An elementary upper bound:
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|P[f](ry)| \leq 2 \|f\|_1 (1 - r) d\sigma(\xi)
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Let \( \beta \in (0, d] \). What is the size of the set of points \( y \) such that \( |P[f](ry)| \approx (1 - r)^{-\beta} \) when \( r \to 1 \)?
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\[ 0 < \beta < d \]

\[ \mathcal{E}(\beta, f) = \left\{ y \in S_d; \limsup_{r \to 1} \frac{|P[f](ry)|}{(1 - r)^{-\beta}} = +\infty \right\} \]
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- For any \( f \in L^1(S_d) \), \( \dim_H (E(\beta, f)) \leq d - \beta \).
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- If $E \subset S_d$ is such that $\dim_H(E) < d - \beta$, there exists $f \in L^1(S_d)$ such that $E \subset \mathcal{E}(\beta, f)$.
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The first part was already obtained by Armitage (1981) in the context of the half upper space.
A more precise result

Let $\tau$ be a nonnegative nonincreasing function such that

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\tau(s) \approx \tau(2s), \quad \lim_{s \to 0^+} \tau(s) = +\infty \quad \text{and} \quad \tau(s) \ll s^{-d}.
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and the gauge function $\phi$ by $\phi(s) = \tau(s)s^d$. 

\[ \text{Theorem (Bayart, H.)} \]

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The Hardy-Littlewood maximal inequality

\[ P[\mu](x) = \int_{S_d} P(x, \xi) \, d\mu(\xi) \]
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where \( \kappa(y, \delta) = \{ \xi \in S_d; \| \xi - y \| < \delta \} \).

\[ \sup_{r \in (0,1)} |P[\mu](ry)| \leq \sup_{\delta > 0} \frac{|\mu(\kappa(y, \delta))|}{\sigma(\kappa(y, \delta))} \]

\( \kappa \) is called a cap.
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Lemma (a quantitative improvement)

Let \( 0 < r < 1 \). There exists \( \delta \geq 1 - r \) such that

\[
|P[\mu](ry)| \leq C \frac{|\mu| (\kappa(y, \delta))}{\sigma(\kappa(y, \delta))},
\]

where \( C \) is a constant independent of \( \mu, r \) and \( y \).
Dimension of $\mathcal{E}(\beta, \mu) :$ the upper bound

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**Dimension of $\mathcal{E}(\beta, \mu)$: the upper bound**

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Let $y \in \mathcal{E}_M$. Using the previous lemma, we can find $r_y$ as close to 1 as we want and a cap $\kappa_y = \kappa(y, \delta_y)$ with $\delta_y \geq 1 - r_y$

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$\delta_y$ goes to 0 when $r_y$ goes to 1.
Dimension of $\mathcal{E}(\beta, \mu)$: the upper bound

\[(1 - r_y)^{-\beta} \sigma(\kappa_y) < \frac{C}{M} |\mu|(\kappa_y).\]
Dimension of $\mathcal{E}(\beta, \mu) : \text{the upper bound}$

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By the Vitali covering lemma, we can find a family of disjoint caps $(\kappa_{y_j})_{j \in \mathbb{N}}$ such that $\mathcal{E}_M \subset \bigcup_i 5\kappa_{y_i}$. 

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$$\mathcal{H}^{d-\beta}(\mathcal{E}_M) \leq \frac{C}{M} \|\mu\|$$

$$\mathcal{H}^{d-\beta}(\mathcal{E}(\beta, \mu)) = 0$$
Lower bound for the dimension: an elementary lemma

If $r > 1/2$, \[ \int_{\kappa(N,1-r)} P(rN, \xi) d\sigma(\xi) \geq C \]
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Lower bound for the dimension: the construction

Let $E$ be such that $\mathcal{H}^{d-\beta}(E) = 0$. Let $\mathcal{R}_j$ be a $2^{-j}$-covering of $E$ by caps such that

$$\sum_{\kappa \in \mathcal{R}_j} |\kappa|^{d-\beta} \leq 2^{-j}.$$

Choose $(\omega_n)_{n \geq 1}$ tending to infinity such that

$$\sum_{n \geq 1} \omega_n \sum_{\kappa \in C_n} |\kappa|^{d-\beta} < +\infty.$$
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$$\sum_{\kappa \in \mathcal{R}_j} |\kappa|^{d-\beta} \leq 2^{-j}.$$ 

Define

$$\mathcal{C}_n = \left\{ \kappa \in \bigcup_{j} \mathcal{R}_j; \ 2^{-(n+1)} < |\kappa| \leq 2^{-n} \right\}.$$
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Choose \( (\omega_n)_{n \geq 1} \) tending to infinity such that

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Observe that $E \subset \lim \sup_n E_n$ where $E_n = \bigcup_{\kappa \in C_n} \kappa$. 
Lower bound for the dimension: the function $f$

\[ f = \sum_{n \geq 1} \omega_n 2^{-n\beta} \sum_{\kappa \in C_n} \mathbb{1}_{4\kappa} \]
Lower bound for the dimension: the function $f$

$$f = \sum_{n \geq 1} \omega_n 2^{-n\beta} \sum_{\kappa \in C_n} \| \mathbb{1}_{4\kappa} \|$$

$$\| f \|_1 \leq C \sum_{n \geq 1} \omega_n 2^{-n\beta} \sum_{\kappa \in C_n} |\kappa|^d \leq C \sum_{n \geq 1} \omega_n \sum_{\kappa \in C_n} |\kappa|^{d-\beta} < +\infty.$$
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\[ P[f](ry) \geq \omega_n 2^{-n\beta} \int_{4\kappa_0} P(ry, \xi) d\sigma(\xi) \]
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The divergence index
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Let \( f \in L^1(S_d) \) and \( y_0 \in S_d \).

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\beta(y_0) = \inf \left( \beta ; |P[f](ry_0)| = O((1 - r)^{-\beta}) \right)
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Level sets :

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Spectrum of singularities :

\[
\beta \mapsto \dim_H(E(\beta, f)).
\]
Multifractal behavior of $P[f]$

Of course, $E(\beta, f) \subset \bigcap_{\gamma < \beta} E(\gamma, f)$, so that

$$\dim_{\mathcal{H}} (E(\beta, f)) \leq d - \beta.$$
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**Theorem (Bayart, H.)**

*For quasi-all functions* $f \in L^1(S_d)$,*

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- For such $f$ we also have $\dim_H(\mathcal{E}(\beta, f)) = d - \beta$. 
The analogue of dyadic numbers in the sphere $S_d$
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There exists a sequence $(\mathcal{R}_n)_{n \geq 1}$ of finite subsets of $S^d$ satisfying

- $\mathcal{R}_n \subset \mathcal{R}_{n+1}$;
- $\bigcup_{x \in \mathcal{R}_n} \kappa(x, 2^{-n}) = S_d$;
- $\text{card}(\mathcal{R}_n) \leq C2^{nd}$;
- For any $x, y$ in $\mathcal{R}_n$, $x \neq y$, then $|x - y| \geq 2^{-n}$. 

If $\alpha > 1$, let $N_n,\alpha = \left\lfloor \frac{n}{\alpha} \right\rfloor + 1$ and $D_{n,\alpha} = \bigcup_{x \in \mathcal{R}_{N_n,\alpha}} \kappa(x, 2^{-n})$.

Proposition $\text{Hd}/\alpha(\limsup_{n \to +\infty} D_{n,\alpha}) = +\infty$.

Proof: mass transference principle.

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**In the way of saturating functions**

\[ f_n := \frac{1}{n+1} \sum_{N=1}^{n+1} \sum_{x \in \mathcal{R}_N} 2^{(n-N)d} \mathbb{1}_{K(x, 2^{2-n})}. \]
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**Proposition**

\( f_n \in L^1(S_d) \) and \( \|f_n\|_1 \leq C. \)

Moreover, for any \( \alpha > 1 \), for any \( y \in D_{n,\alpha} \),

\[ P[f_n](r_n y) \geq \frac{C}{n} 2^{(n-N_{n,\alpha})d}, \]

where \( 1 - r_n = 2^{-n} \), \( N_{n,\alpha} = \lceil n/\alpha \rceil + 1 \) and \( C \) is independent of \( n \) and \( \alpha \).
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Remark: \( 2^{(n-N_{n,\alpha})d} \approx (1 - r_n)^{-\beta} \) if \( \frac{d}{\alpha} = d - \beta \).
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Using the positivity of the Poisson kernel, we get

\[ P[f_n](r_n y) \geq \frac{1}{n+1} \int_{\kappa(y, 2^{-n})} 2^{(n-N_{n,\alpha})d} P(r_n y, \xi) d\sigma(\xi) \]
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Construction of a dense sequence

Proposition

There exists a dense sequence \((h_n)_{n \geq 1}\) in \(L^1(S_d)\) such that for any \(n \geq 1\), for any \(\alpha > 1\) and any \(y \in D_{n,\alpha}\),

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Let \((g_n)_{n \geq 1}\) be a dense sequence of continuous functions such that \(\|g_n\|_\infty \leq n\).

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\(\|P[g_n]\|_{\infty} \leq n\). Then, \(P[h_n](r_n y) \geq \frac{1}{n} P[f_n](r_n y) - n\).
The dense $\mathcal{G}_\delta$ set

The residual set we will consider is the dense $G_\delta$-set

$$A = \bigcap_{k \geq 1} \bigcup_{n \geq k} B_{L^1}(h_n, \delta_n).$$

where $\delta_n$ is such that $\|f\|_1 \leq \delta_n \Rightarrow \|P[f](r_n \cdot)\|_\infty \leq 1.$
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$$\frac{\log |P[f](r_n y)|}{-\log(1 - r_n)} \geq \left( d - \frac{N_{n,\alpha}d}{n} \right) + o(1).$$

$$d - \frac{N_{n,\alpha}d}{n} \approx d - \frac{d}{\alpha} = \beta \quad \text{if} \quad \frac{d}{\alpha} = d - \beta.$$
The case of nonnegative harmonic functions

The set $\mathcal{H}^+(B_{d+1})$ of nonnegative harmonic functions in the ball $B_{d+1}$ endowed with the topology of the locally uniform convergence is a closed cone in the space of all continuous functions in the ball: it satisfies Baire’s property.
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**Theorem**

*For quasi-all nonnegative harmonic functions $h$ in the unit ball $B_{d+1}$, for any $\beta \in [0, d]$,*

$$\dim_{\mathcal{H}} (E(\beta, h)) = d - \beta$$

*where*

$$E(\beta, h) = \left\{ y \in S_d ; \limsup_{r \to 1} \frac{\log h(ry)}{-\log(1-r)} = \beta \right\}.$$
Merci !