Generic boundary behaviour for harmonic functions in the ball

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Domaine du Rond-Chêne, May 2015

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- $r \to 1$ corresponds to the radial convergence in the disk.

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- Hunt and Wheeden (1970) : If h is a nonnegative harmonic function in a Lipschitz domain $U \subset \mathbb{R}^n$, then h has a non tangential limit at almost every point of the boundary ∂U.

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The first part was already obtained by Armitage (1981) in the context of the half upper space.

A more precise result

Let τ be a nonnegative nonincreasing function such that

$$\tau(s) \approx \tau(2s), \quad \lim_{s \to 0^+} \tau(s) = +\infty \quad \text{and} \quad \tau(s) \ll s^{-d}.$$

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Lemma (a quantitative improvement)

Let $0 < r < 1$. There exists $\delta \geq 1 - r$ such that

$$\left|P[\mu](ry)\right| \leq C\frac{|\mu|(\kappa(y,\delta))}{\sigma(\kappa(y,\delta))},$$

where C is a constant independent of μ, r and y.

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δ_y goes to 0 when r_y goes to 1.

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$$\boxed{\mathcal{H}^{d-\beta}(\mathcal{E}(\beta,\mu)) = 0}$$

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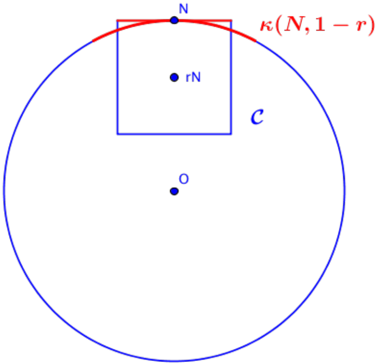

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Let E be such that $\mathcal{H}^{d-\beta}(E)=0$. Let $\mathcal{R}_j$ be a 2^{-j}-covering of E by caps such that

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$$f = \sum_{n\geq 1} \omega_n 2^{-n\beta} \sum_{\kappa \in \mathcal{C}_n} \mathbb{1}_{4\kappa}$$

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Multifractal behavior of $P[f]$

Of course, $E(\beta, f) \subset \bigcap_{\gamma<\beta} \mathcal{E}(\gamma, f)$, so that

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- For such f we also have $\dim_{\mathcal{H}} \left(\mathcal{E}(\beta, f)\right) = d - \beta$.

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Remark : we can replace n by a subsequence n_k.

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$$f_n := \frac{1}{n+1} \sum_{N=1}^{n+1} \sum_{x \in \mathcal{R}_N} 2^{(n-N)d} \mathbb{1}_{\kappa(x, 2 \cdot 2^{-n})}.$$

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Construction of a dense sequence

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There exists a dense sequence $(h_n)_{n\geq 1}$ *in* $L^1(\mathcal{S}_d)$ *such that for any* $n \geq 1$, *for any* $\alpha > 1$ *and any* $y \in D_{n,\alpha}$,

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The dense $\mathcal{G}_\delta$ set

The residual set we will consider is the dense G_δ-set

$$A = \bigcap_{k\geq 1} \bigcup_{n\geq k} B_{L^1}(h_n, \delta_n).$$

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$d - \frac{N_{n,\alpha}d}{n} \approx d - \frac{d}{\alpha} = \beta \quad \text{if} \quad \frac{d}{\alpha} = d - \beta.$

The case of nonnegative harmonic functions

The set $\mathcal{H}^+(B_{d+1})$ of nonnegative harmonic functions in the ball B_{d+1} endowed with the topology of the locally uniform convergence is a closed cone in the space of all continuous functions in the ball : it satisfies Baire's property.

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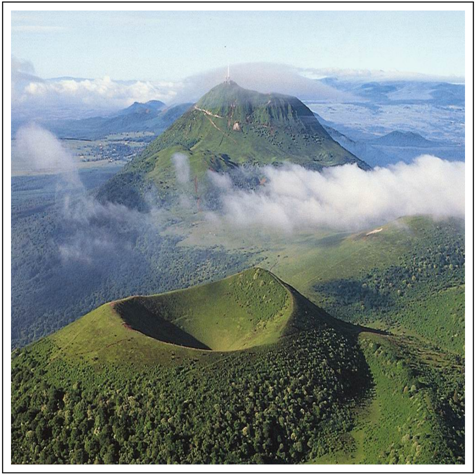
Theorem

For quasi-all nonnegative harmonic functions h in the unit ball B_{d+1}*, for any* $\beta \in [0, d]$*,*

$$\dim_{\mathcal{H}}\big(E(\beta, h)\big) = d - \beta$$

where

$$E(\beta, h) = \left\{ y \in \mathcal{S}_d \ ; \ \limsup_{r \to 1} \frac{\log h(ry)}{-\log(1-r)} = \beta \right\}.$$



Merci !