Lowest fractal dimensions for universal differentiability

Olga Maleva
University of Birmingham, UK

Genericity and Small sets in Analysis
Esneux
May 28, 2015

Background

We consider real-valued Lipschitz functions $f : X \to \mathbb{R}$.

- If X is finite-dimensional, then there is a *Rademacher theorem* which implies f is differentiable almost everywhere w.r.t. the Lebesgue measure.
  - If $A \subset X$ is a set of positive measure and X is finite-dimensional, then $\{x \in A : f \text{ is differentiable at } x\}$ is not empty.

1. What if A has measure 0?
2. What if X is infinite-dimensional?

1. For infinite-dimensional separable X, the dual X^* must be separable as otherwise there is an equivalent norm on X which is everywhere Fréchet non-differentiable.

2. X^* separable $\implies$ every Lipschitz function is differentiable on a dense subset of X [Preiss, 1990] and ...

...moreover, points of differentiability can be found inside *any* fixed beforehand dense G_δ subset S of X satisfying the condition that S contains a dense set of lines.

Universal Differentiability Set (UDS)

A Borel set $S \subseteq X$ is a UDS if for every Lipschitz function $f : X \to \mathbb{R}$ there is an $x \in S$ such that f is (Fréchet) differentiable at x.

Background

Classical results: null subsets of $\mathbb{R}$

Any null subset of $\mathbb{R}$ is a non-UDS.
Proof: Let $|E| = 0$, $E \subset \mathbb{R}$. Choose open sets G_n s.t. $|G_1| < \infty$, $\mathbb{R} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots \supset E$ and

$|G_{n+1} \cap C| \leq \frac{1}{2^{n+1}}|C|$ for every component C of G_n, $n \geq 1$.

Let $\phi(t) = (-1)^n$ if $t \in G_n \setminus G_{n+1}$; ϕ is defined *a.e.* on $\mathbb{R}$ (for $t \notin \bigcap G_n$).

The function $f(x) = \displaystyle\int_0^x \phi(t)dt : \mathbb{R} \to \mathbb{R}$ is Lipschitz with the Lipschitz constant 1.

Assume $x \in E$ and $(a, b) = C$ is the component of G_n containing x.
Then $|C \cap G_{n+1}| \leq \frac{1}{2^{n+1}}(b - a)$ and

$$\left|\frac{f(b) - f(a)}{b - a} - (-1)^n\right| \leq \frac{1}{b-a}\int_a^b |\phi(t) - (-1)^n|dt$$

$$= \frac{1}{b-a}\int_{C \cap G_{n+1}} |\phi(t) - (-1)^n|dt \leq \frac{2|C \cap G_{n+1}|}{b-a} \leq \frac{1}{2^n}.$$

Results in dimensions $n \geq 2$

If $n \geq 2$ one can choose a G_δ set $S \subseteq \mathbb{R}^n$ to contain all rational lines and to have measure 0. Hence there are *Lebesgue null* universal differentiability subsets of $\mathbb{R}^n$, $n \geq 2$.

X is infinite-dimensional $\implies$ replacing rational lines with lines going in directions of a countable dense subset of X, we get G_δ sets which are examples of UDS in infinite-dimensional spaces.

1. The closure of the set constructed by Preiss is always equal to the whole space.

2. M. Doré–O.M. (2010 + 2011): $n \geq 1 \implies \mathbb{R}^n$ contains a compact UDS of Hausdorff dimension 1 (so its Lebesgue measure is zero if $n \geq 2$).

3. M. Doré–O.M. (2012): X^* separable $\implies$ there is a closed bounded totally disconnected UDS of Hausdorff dimension 1.

4. M. Dymond–O.M. (2013): In any $\mathbb{R}^n$ there is a (compact) UDS of Minkowski (box counting) dimension 1 (and it is Hausdorff dim 1 too).

Results in dimensions $n \geq 2$ (cont.)

5. D. Preiss–G. Speight (2013):
$n > m \geq 1$, $\varepsilon > 0 \implies$ there is a set $S \subseteq \mathbb{R}^n$ of Hausdorff dimension less than $m + \varepsilon$ such that it is a UDS w.r.t. Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^m$.

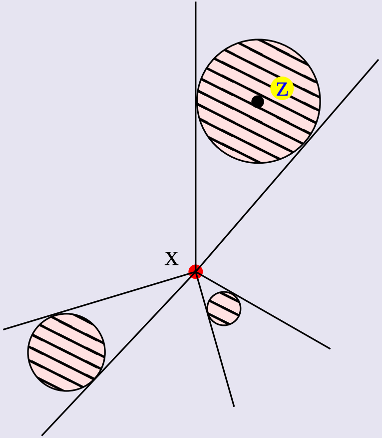
6. G. Alberti, M. Csörnyei, D. Preiss (2010): $n = m = 2 \implies \forall S \subset \mathbb{R}^2$ of Lebesgue measure 0 is a non-UDS w.r.t. Lipschitz $f : \mathbb{R}^2 \to \mathbb{R}^2$.

7. M. Csörnyei–P. Jones (announced): $n = m > 2 \implies \forall S \subset \mathbb{R}^n$ of Lebesgue measure 0 is a non-UDS w.r.t. Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^n$.

UDS: necessary condition

Classical results: porosity implies non UDS

Let $\lambda > 0$. $E \subseteq X$ is λ-porous at $x \in X$ if for every $r > 0$ there is a $z \in B(x,r)$ such that $B(z, \lambda\|z-x\|) \cap E = \emptyset$.



E is porous at $x \in E \Rightarrow$ $f(y) = \mathrm{dist}(y, E)$ is 1-Lipschitz and is not differentiable at x.

$$\frac{f(z) - f(x)}{\|z - x\|} \geq \lambda$$

$E \subseteq X$ is porous if $\exists \lambda > 0$ s.t. it is λ-porous at each of its points.

B. Kirchheim, D. Preiss, L. Zajíček (1980s):
Sigma-porous sets, countable unions of porous sets, are also non UDS.

UDS: sufficient condition

Curve non-porosity implies UDS
Approximation property (Dymond–O.M., 2013)

If $n \geq 2$ and $(E_\lambda)_{\lambda \in (0,1)} \subseteq \mathbb{R}^n$ is an increasing sequence of closed sets satisfying the following *approximation property*:
for all $0 < \lambda < \lambda' < 1$ and $\eta > 0$
there is a threshold $\delta^* = \delta^*(\lambda, \lambda', \eta)$ such that $\forall\, x \in E_\lambda$, $\|e\| = 1$,
$0 < \delta < \delta^* \implies$
there exists $[x', x' + \delta e'] \subseteq E_{\lambda'}$ with $\|x - x'\| < \eta\delta$ and $\|e - e'\| < \eta$,

then each E_λ is a universal differentiability set.

(Dymond 2013)

Moreover, $[x', x' + \delta e'] \subset E_{\lambda'}$ may be replaced by existence of γ, a Lipschitz curve with $\gamma(0) = x'$ and $\gamma'(t) \underset{\eta}{\approx} e'$ s.t.
$\mathcal{H}^1(\gamma \cap E_{\lambda'}) \geq (1 - \eta)\mathcal{H}^1(\gamma)$.

Geometric measure theory

Equivalent definitions of a **u.p.u. set**

Theorem. G. Alberti, M. Csörnyei, D. Preiss (2010): $S \subseteq \mathbb{R}^n$ The following two conditions are equivalent:

1. There exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\forall x \in S$ and $\forall \|e\| = 1$ the directional derivative $f'(x, e)$ does not exist
2. S is C-null for every cone C, i.e.
   for every $C = \{v : \|v - v_0\| < \alpha\}$ and for every $\varepsilon > 0$
   there exists an open set G_ε with $S \subseteq G_\varepsilon$ and
   $$\mathcal{H}^1(\gamma \cap G_\varepsilon) \leq \varepsilon$$
   for every C^1-curve γ whose tangents lie in C.

u.p.u. $\Rightarrow$ p.u.

Uniformly purely unrectifiable (u.p.u.) sets are **purely (1-)unrectifiable**: their intersection with any smooth curve has 1-dimensional measure 0.

Geometric measure theory - AP

u.p.u. $\Rightarrow$ p.u.

Each uniformly purely unrectifiable set is **purely (1-)unrectifiable**: its intersection with any smooth curve has 1-dimensional measure 0.

p.u. $\Rightarrow$ u.p.u. (A. Mathé, 2014)

Each purely (1-)unrectifiable set is **uniformly purely unrectifiable**.

Approximation property

UDS $\Longrightarrow$ not u.p.u. $\Longrightarrow$ not p.u. $\Longrightarrow$ intersections with Lipschitz curves around *kernel points* have positive measure.
Question: is AP necessarily satisfied?
Equivalently, are there UDS with *kernel points* without approximation in some directions or approximations of too short length?

Using AP to find a point of differentiability in a set

Assume we are proving that $E \subseteq X$ is a UDS, i.e.
every Lipschitz function $f : X \to \mathbb{R}$ has a point of differentiability in E.
How do we find a point of differentiability?

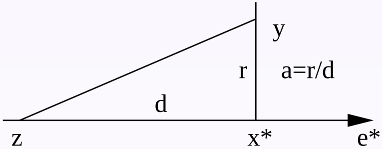
Step by step

We construct a sequence (x_k, e_k), $x_k \in E$ and $\|e_k\| = 1$ such that $f'(x_k, e_k)$ exists and is "almost maximal" among $f'(x, e)$ when $x \in E$, $\|x - x_k\|$ is small and e is arbitrary direction.

$x_k \to x^*$, $e_k \to e^*$ and $f'(x^*, e^*)$ exists, is equal to $\lim f'(x_k, e_k)$ and is therefore "almost maximal" in every neighbourhood of x^*.

We then prove f is differentiable at x^* and $f'(x^*)(u) = f'(x^*, e^*)\langle u, e^* \rangle$.

Finding a point of differentiability

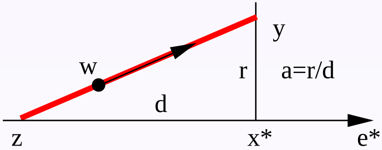


$$M = f'(x^*, e^*) \geq 0$$

$f(y) > f(x^*) + \varepsilon r$
$f(z) \approx f(x^*) - f'(x^*, e^*)d$

$$\frac{f(y) - f(z)}{\|y - z\|} \geq \frac{Md + \varepsilon r}{\sqrt{d^2 + r^2}} = \frac{M + \varepsilon a}{\sqrt{1 + a^2}} > M + \varepsilon a + O(a^2) > M + \tau$$

Finding a point of differentiability



$\frac{f(y)-f(z)}{\|y-z\|} > f'(x^*, e^*) + \tau$, $\tau > 0$ is fixed

Therefore there exists $w \in [y, z]$ such that $f'(w, \frac{y-z}{\|y-z\|}) > f'(x^*, e^*) + \tau$

If $[y, z] \subseteq E$, we get a contradiction.

Thus $f'(x^*, e^{*\perp}) = 0$.

Hausdorff and Minkowski dimension

Hausdorff dimension

$$\mathcal{H}^p(A) = \lim_{\delta \downarrow 0} \inf \Big\{ \sum_i \mathrm{diam}(E_i)^p : A \subseteq \bigcup_i E_i, \mathrm{diam}(E_i) \leq \delta \Big\}.$$

is the p-dimensional Hausdorff measure of A.

Hausdorff dimension:

$$\underline{\dim_{\mathcal{H}}(A) = \inf\{p : \mathcal{H}^p(A) = 0\}}.$$

Minkowski (box counting) dimension

Now for each $\delta > 0$ let N_δ be the minimal possible number of balls of radius δ with which it is possible to cover A. Then

$$\underline{\overline{\dim}_{\mathcal{M}}(A)/\underline{\dim}_{\mathcal{M}}(A) = \inf\{p : \overline{\lim}_{\delta\downarrow 0}/\underline{\lim}_{\delta\downarrow 0} N_\delta \delta^p = 0\}}$$

is the upper (lower) Minkowski dimension of A.

Properties of UDS

E UDS, f Lipschitz $\implies |P(E)| \geq |P(E \cap D_f)| > 0$ for all $P \in X^* \setminus \{0\}$

Theorem (Zahorski '46, Fowler-Preiss '09): Given any G_δ set $G \subseteq \mathbb{R}$ of measure zero, there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant 1, which is differentiable everywhere outside G and for any $x \in G$, $g'_\pm(x) = \pm 1$.

Assume $|P(E \cap D_f)| = 0$, let $G \supseteq P(E \cap D_f)$ be a G_δ set of measure 0. Let $Pe = 1$ and $F := f + 2\|e\|\mathrm{Lip}(f) g \circ P$. Then $E \cap D_F = \emptyset$.

(weak Projection property)

E is a UDS $\implies \overline{\mathbf{dim}}_{\mathcal{M}}(E) \geq \underline{\mathbf{dim}}_{\mathcal{M}}(E) \geq \mathbf{dim}_{\mathcal{H}}(E) \geq 1$

Assume $\dim_{\mathcal{H}}(E) < 1$; let $P \in X^* \setminus \{0\}$.

$\dim_{\mathcal{H}}(P(E)) < 1 \quad \Rightarrow \quad S = P(E) \subseteq \mathbb{R}$ is Lebesgue null.

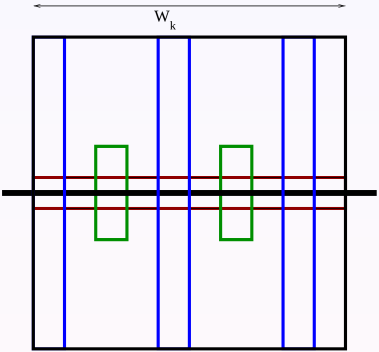
If $\overline{\dim}_{\mathcal{M}}(E) = 1$ and E is a UDS then $\dim_{\mathcal{M}}(E) = \dim_{\mathcal{H}}(E) = 1$.

E is a UDS $\implies \mathcal{H}^1(E) = \infty$, even not σ-finite

Projection property - weak and strong

1. Weak PP:
E UDS, f Lipschitz $\implies |P(E)| \geq |P(E \cap D_f)| > 0$ for all $P \in X^* \setminus \{0\}$.
2. Strong PP:
If E satisfies curve approximation property then
$P(D \cap B(x, r))$ has a full measure on $(Px - \Delta, Px + \Delta)$, $\Delta > 0$.

Construction in the finite-dimensional case (Dymond-O.M., 2013)



$R = R_{k+1} = Q^s,\ Q > 1,\ w_{k+1} = w_k/R$

Total number of cubes $w_{k+1} \times w_{k+1}$:

0. R red cubes
1. $R + a \times Q^s + aQ \times Q^{s-1} + \cdots +$
   $+ aQ^{s-1} \times Q \sim asQ^s = aR \log R$

$\mapsto$ in 'many directions' $abR \log R$

2. Repeat for $\forall$ new tube $\implies R(ab \log R)^2$
3. Again and again: $R(ab \log R)^m$ cubes.

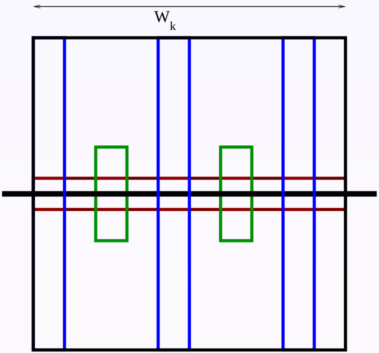
$N_{w_{k+1}} \leq N_{w_k} \times R(c \log R)^m$

As $p > 1$,
$\frac{N_{w_{k+1}} w_{k+1}^p}{N_{w_k} w_k^p} \leq (c \log R)^m R^{1-p} < 1$, R large

For $\delta \in (w_{k+1}, w_k)$: $N_\delta \delta^p \leq N_{w_{k+1}} w_k^p = N_{w_{k+1}} w_{k+1}^p R^p$.

We show: $N_{w_{k+1}} w_{k+1}^p R_{k+1}^p \to 0$

Further ideas (Dymond-O.M., 2014)



Can we get $N_{w_{k+1}} \le N_{w_k} \times R\Phi(R)$ for *any* $\Phi(R) \nearrow \infty$ chosen in advance?

Describe the class of gauge functions f for which
$\overline{\mathcal{M}_f}(N) = \overline{\lim}_{\delta>0} N_\delta f(\delta)$ or
$\underline{\mathcal{M}_f}(N) = \underline{\lim}_{\delta>0} N_\delta f(\delta)$
is finite.

We know $\lim_{\delta\to 0} N_\delta \delta$ is infinite $\forall$ UDS but it's possible $N_\delta \delta^p \to 0 \; \forall \; p > 1$.

Question

Can we find UDS with dimension function strictly between x and x^p, $p > 1$?
Conjecture: Yes, but not below $x/\log x$.

Open questions

Typical behaviour on non-UDS

Recall Alberti, Csörnyei, Preiss and Jones, Csörnyei show
$\forall E \subset \mathbb{R}^n$ null sets there exists $f : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz which is nowhere differentiable on E.
Are some sets better than other?

D. Preiss–J. Tišer (1995): Let $E \subset [0, 1]$. A typical Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at no point of E if and only if E is contained in a null F_σ subset of $[0, 1]$.
Careful: In case $\mathbb{R}^n \to \mathbb{R}$ mappings there are closed UDS!