When is a positive cone Haar null?

(joint with J. Esterle and P. Moreau)

All Banach spaces are real and infinite-dimensional.

Haar null sets

Thanks to Frédéric, you all know the definition: a Borel set A in a separable Banach space X is Haar null if there exists a Borel probability measure μ on X such that

$$\mu(A + x) = 0 \quad \text{for all } x \in X \,.$$

Starting point

Matouskova–Stegall: Haar null sets characterize reflexivity. Precisely: *A separable Banach space X is reflexive if and only if every closed convex set $C \subseteq X$ with empty interior is Haar null.*

Vague question. What about "special" closed convex sets with empty interior in nonreflexive spaces?

Positive cones

$$\mathbf{e} = (e_i)_{i\geq 1} \text{ basic sequence in } X$$

$$Q^+(\mathbf{e}) := \left\{ \sum_{i=1}^{\infty} x_i\, e_i ;\; x_i \geq 0 \right\}$$

(Closed convex set with empty interior in $[\mathbf{e}] := \overline{\mathrm{span}}\,\{e_i;\, i \geq 1\}$.)

Precise question. When is $Q^+(\mathbf{e})$ Haar null in $[\mathbf{e}]$?

Remark. $Q^+(\mathbf{e})$ is never *Gauss null*.

Two simple examples

Example 1. The positive cone of ℓ_1 is Haar null.

Proof. Let $(\xi_i)_{i\geq 1}$ be a sequence of independent (real) random variables with $\mathbb{P}(\xi_i = 0) = 1 - \frac{1}{i}$ and $\mathbb{P}(\xi_i = -\frac{1}{i}) = \frac{1}{i}$. Then $\|\xi_i\|_{L_1} = \frac{1}{i^2}$, so that $\xi := (\xi_1, \xi_2, \dots) \in \ell_1$ almost surely. Let μ be the distribution of the random variable ξ (measure on ℓ_1). If $x = (x_i) \in \ell_1$ and $x \leq 0$, then

$$\mu(Q^+ + x) = \mathbb{P}(\xi \geq x) = \prod_{i\in\mathbb{N}} \mathbb{P}(\xi_i \geq x_i) = \prod_{\{i;\, x_i > -1/i\}} \left(1 - \tfrac{1}{i}\right) = 0\,.$$

Example 2. The positive cone of c_0 contains a translate of every compact set, so it is not Haar null.

Proof. If $K \subseteq c_0$ is compact, then it is "uniformly c_0" :

$$\alpha_i := \sup_{x=(x_j)\in K} \sup_{j\geq i} |x_j| \xrightarrow{i\to\infty} 0\,.$$

So $\alpha := (\alpha_i) \in c_0$, and $K + \alpha \subseteq Q^+$.

A tempting "conjecture"

Assume that $\mathbf{e}$ is a normalized and *unconditional* basic sequence. Then $Q^{+}(\mathbf{e})$ is not Haar null if and only if $\mathbf{e}$ is equivalent to the canonical basis of c_0.

A remark to keep in mind

$$\mathbf{e} = (e_i) \text{ basic sequence}$$

• To show that $\mathbf{e} \sim c_0$ means to get estimates of the form

$$c\,\|a\|_\infty \le \left\|\sum_i a_i e_i\right\| \le C\,\|a\|_\infty \quad \text{for } a = (a_i) \in c_{00}.$$

• If $\mathbf{e}$ is normalized, the *lower* estimate is *for free*. So one just need to check the upper estimate

$$\left\|\sum_i a_i e_i\right\| \le C\,\|a\|_\infty .$$

A weaker version of the conjecture

Say that a set $Q \subseteq X$ is a "compact-catcher" if it contains a translate of every compact set.

Theorem 1. *The conjecture holds true if "not Haar null" is replaced with "compact-catcher": if* $\mathbf{e}$ *is a normalized unconditional basic sequence such that* $Q^+(\mathbf{e})$ *is a compact-catcher, then* $\mathbf{e} \sim c_0$*.*

Remark. False without the unconditionality assumption. Example: the canonical basis of the *James' space* has a compact-catcher positive cone.

Proof of Theorem 1 (sketch)

Key lemma 1. *Assume that* $\mathbf{e} = (e_i)$ *is normalized. Then,* $Q^+(\mathbf{e})$ *is a compact-catcher if and only if one can find a sequence* (λ_i) *with* $\lambda_i \geq 1$ *such that*

$$\sup_{n \geq 1} \left\| \sum_{i=1}^{n} \lambda_i e_i \right\| < \infty.$$

Key lemma 1 + unconditionality $\implies$ upper estimate:

$$\left\| \sum_i a_i e_i \right\| = \left\| \sum_i \frac{a_i}{\lambda_i} \lambda_i e_i \right\| \leq K \, \|a\|_\infty \left\| \sum_i \lambda_i e_i \right\| \leq C \, \|a\|_\infty.$$

Proof of Key lemma 1 (only if part)

Fact. *If $Q^+(\mathbf{e})$ is a compact-catcher, one can find $M < \infty$ such that the following holds: for any finite set $F \subseteq \overline{B}(0,1)$, there exists $z \in X$ such that $\|z\| \leq M$ and $z + F \subseteq Q^+(\mathbf{e})$.*

Now, define

$$F_k := \{-e_i;\ 1 \leq i \leq k\} \qquad (k \geq 1).$$

Fact $\implies$ there exists $z_k = \sum_i z_{i,k} e_i$ such that $\|z_k\| \leq M$ and $z_k + F_k \subseteq Q^+(\mathbf{e})$.

- $\|z_k\| \leq M \implies$ the $z_{i,k}$ are bounded ($\mathbf{e}$ normalized);
- $z_{i,k} \geq 1$ if $k \geq i$.

WLOG $z_{i,k} \xrightarrow{k\to\infty} \lambda_i \geq 1$ for all $i \in \mathbb{N}$; and then

$$\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| = \lim_{k\to\infty} \left\| \sum_{i=1}^{n} z_{i,k} e_i \right\| \leq KM \quad \text{for all } n \in \mathbb{N}.$$

Block sequences

Block sequence of $\mathbf{e} = (e_i)$: sequence $\mathbf{f} = (f_j)_{j \geq 1}$ ¿of the form

$$f_j = \sum_{i=p_{j-1}}^{p_j - 1} a_i e_i \qquad (1 \leq p_0 < p_1 < \dots)$$

Block sequence lemma. *Assume that* $\mathbf{e}$ *is unconditional. If* $\mathbf{e}$ *has a block sequence* $\mathbf{f}$ *such that* $Q^+(\mathbf{f})$ *is Haar null in* $[\mathbf{f}]$*, then* $Q^+(\mathbf{e})$ *is Haar null in* $[\mathbf{e}]$*.*

Consequence. If $Q^+(\mathbf{e})$ is not Haar null, then every block sequence of $\mathbf{e}$ has a further block sequence equivalent to the canonical basis of c_0. (Follows from Matouskova–Stegall + James' characterization of reflexivity.)

c_0-saturation of block sequences

e unconditional

Theorem 2. *If $Q^+(\mathbf{e})$ is not Haar null, then every normalized block-sequence of* **e** *has a subsequence equivalent to the canonical basis of c_0 (with uniform bounds on the implied constants).*

Remark. Something stronger holds true. But still not enough to ensure that $\mathbf{e} \sim c_0$.

Proof of Theorem 2 (sketch)

Enough: to show that **e** itself has a c_0 subsequence (assuming **e** is normalized and unconditional, with $Q^+(\mathbf{e})$ not Haar null).

Key lemma 2. *One can find $\delta > 0$ and $C < \infty$ such that the following holds: for every finite set $I \subseteq \mathbb{N}$, there exists $J \subseteq I$ such that*

$$|J| \geq \delta\, |I| \quad \textit{and} \quad \Big\| \sum_{i \in J} e_i \Big\| \leq C\,.$$

Define

$$\mathcal{J} := \Big\{ J \subseteq \mathbb{N} \text{ finite};\ \big\| \textstyle\sum_{i \in J} e_i \big\| \leq C \Big\}\,.$$

Key lemma 2 + Ptak's Lemma $\implies$ there exists an infinite set $\mathbb{I} \subseteq \mathbb{N}$ such that every finite subset of $\mathbb{I}$ belongs to $\mathcal{J}$.

Unconditionality $\implies$ upper estimate for $(e_i)_{i \in \mathbb{I}}$ (as before).

Proof of Key lemma 2

Fact. *Assuming that* $Q^+(\mathbf{e})$ *is not Haar null, one can find* $\delta > 0$ *and* $R < \infty$ *such that the following holds: for every probability measure* μ *supported on* $\overline{B}(0,1)$*, there exists* $x \in X$ *such that* $\|x\| \le R$ *and* $\mu\big(x + Q^+(\mathbf{e})\big) \ge \delta$*.*

Now, $I \subseteq \mathbb{N}$ finite. Apply the Fact with

$$\mu := \frac{1}{|I|} \sum_{i \in I} \delta_{-e_i} \, .$$

This gives $x = \sum_1^\infty x_i e_i$ with $\|x\| \le R$ such that

$$|J| \ge \delta\, |I| \quad \text{where} \quad J = \{i \in I;\, x_i \le -1\}.$$

Then $\left\|\sum_{i \in J} e_i\right\| \le K \,\|x\| \le KR := C$ by unconditionality.

c_0-saturation of quotients

Theorem 3. *Let X be a Banach space, and assume that X has an unconditional basis whose positive cone is not Haar null. Then any quotient E of X is c_0-saturated: every infinite-dimensional subspace of E has a further subspace isomorphic to c_0.*

Remark 1. Again, not enough to prove the conjecture. Example: the *Schreier space* $\mathcal{S}$ has c_0-saturated quotients (Odell 1992).

Remark 2. Little to do with Haar null sets. Say that a Banach space X has Property (P) if every weakly null normalized sequence in X has a subsequence equivalent to the canonical basis basis of c_0. Key lemma 3: *if X has (P) and admits an unconditional basis* $\mathbf{e}$ *with no ℓ_1 block sequence, then every quotient of X has (P).*