Lineability: The search for linearity in Mathematics

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genericity & small sets in analysis

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Stéphane Jaffard's birthday party!

1992

2015

Happy Birthday, Stéphane!

Lineability...? Motivation!

In 1872, **Weierstrass** constructed a continuous nowhere differentiable function on $\mathbb{R}$.
A function such as

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(3^n x)}{2^n}$$

enjoys this property.
In the literature, this example is known as

Weierstrass' monster,

although earlier (1822) **Bolzano** found a function of this type!!!!

How many examples are like Weierstrass'?

MANY functions enjoying this "pathological" property have been constructed since the 1800's. Moreover:

- **1966**: Gurariy showed that there exists an infinite dimensional linear space every nonzero function of which is continuous and nowhere differentiable on $\mathbb{R}$.
- **1999**: Fonf, Gurariy, and Kadeč proved that the above space can be chosen to be **closed** in $\mathcal{C}[0,1]$.
- Analogously, Rodríguez-Piazza (**1995**), Hencl (**2000**), Bayart and Quarta (**2007**), among others, have improved these spaces by adding *extra pathologies* to Weierstrass' monster.

Definitions

Gurariy's results from 1966 and 1999 lead to the introduction of the following concept:

Definition (Gurariy)

- A subset M of functions on $\mathbb{R}$ is said to be **spaceable** if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace.
- The set M will be called **lineable** if $M \cup \{0\}$ contains an infinite dimensional vector space.
- M is called κ**−lineable** if it contains a vector space of dimension κ.

Thus,

Theorem (Gurariy, 1966)

The set of continuous nowhere differentiable functions in $\mathbb{R}$ *is* LINEABLE.

Theorem (Fonf, Gurariy, Kadeč, 1999)

The set of continuous nowhere differentiable functions on $\mathcal{C}[0,1]$ *is spaceable.*

And, also, in 1966 V. I. Gurariy showed something really surprising:

Doklady 1966
Tom 167, No. 5

SUBSPACES AND BASES IN SPACES OF CONTINUOUS FUNCTIONS

V. I. GURARIĬ

Theorem 10. *If all elements of a subspace* E *of* C *are differentiable on* [0, 1], *then* E *is finite-dimensional.**

The term lineability was coined by **Vladimir I. Gurariy** and first introduced by Aron, Gurariy, S. in

Proc. Amer. Math. Soc. **133** *(2004) 795–803.*

Around that time and since then, many authors have shown their interest in this topic...

Other examples...

Theorem (Aron, Gurariy, Seoane, 2004)

The set of differentiable nowhere monotone functions on $\mathbb{R}$ is $\aleph_0$-lineable.

Theorem (Gámez, Muñoz, Sánchez, Seoane, 2010)

The set of differentiable nowhere monotone functions on $\mathbb{R}$ is $\mathfrak{c}$-lineable.

Set of zeroes of polynomials in Banach spaces

- Aron, Rueda (1997).
- Plichko, Zagorodnyuk (1998).
- Aron, Gonzalo, Zagorodnyuk (2000).
- Aron, García, Maestre (2001).
- Aron, Boyd, Ryan, Zalduendo (2003).
- Aron, Hajék (2006).

Chaos and hypercyclicity

- Godefroy, Shapiro (1991).
- Montes (1996).
- Aron, García, Maestre (2001).
- Aron, Bès, León, Peris (2005).
- Seoane (2007).
- Aron, Conejero, Peris, Seoane (2007).
- Bernal (2009).
- Shkarin (2010).
- Bertoloto, Botelho, Fávaro, Jatobá (2012).

Continuous nowhere differentiable functions in $\mathcal{C}[0,1]$

- Rodríguez-Piazza (1995).
- Fonf, Gurariy, Kadeč (1999).
- Aron, García, Maestre (2001).
- Bayart, Quarta (2007).
- Bernal (2008).
- Aron, García, Pérez, Seoane (2009).

Norm-attaining functionals

- Aron, García, Maestre (2001).
- Acosta, Aizpuru, Aron, García (2007).
- Pellegrino, Teixeira (2009).

Subsets of $\mathbb{R}^{\mathbb{R}}$

- Aron, Gurariy, Seoane (2004).
- Enflo, Gurariy (2004).
- Gurariy, Quarta (2004).
- Bayart, Quarta (2007).
- Aron, Seoane (2007).
- Aron, Gorkin (2007).
- García, Palmberg, Seoane (2007).
- Aizpuru, Pérez, García, Seoane (2008).
- Azagra, Muñoz, Sánchez, Seoane (2009).
- Aron, García, Pérez, Seoane (2009).
- Gámez, Muñoz, Seoane (2010, 2011).
- Bartoszewicz, Głąb, Pellegrino, Seoane-Sepúlveda (2012).
- Jimenez-Rodríguez, Muñoz, Seoane (2012).
- Conejero, Jimenez-Rodríguez, Muñoz, Seoane (2012).
- Enflo, Gurariy, Seoane (2014).

ℓ_p and L_p spaces

- Aizpuru, Pérez, Seoane (2005).
- Aron, Pérez, Seoane (2006).
- Muñoz, Palmberg, Puglisi, Seoane (2008).
- Botelho, Diniz, Fávaro, Pellegrino (2011).
- Bernal, Ordóñez-Cabrera (2012).
- Botelho, Fávaro, Pellegrino, Seoane (2012).
- Botelho, Cariello, Fávaro, Pellegrino, Seoane (2012).
- Akbarbaglu, Maghsoudi (2012).
- Jimenez-Rodríguez, Maghsoudi, Muñoz (2013).
- Cariello, Seoane (2013).

Theory of homogeneous polynomials

- Botelho, Matos, Pellegrino (2009).

Complex analysis and holomorphy

- Bernal (2008).
- López (2010).
- Bastin, Conejero, Esser, Seoane (2013).

Measurable and non-measurable functions

- García, Seoane (2006).
- Muñoz, Palmberg, Puglisi, Seoane (2008).

Non-absolutely summing operators

- Puglisi, Seoane (2008).
- Botelho, Diniz, Pellegrino (2009).
- Kitson, Timoney (2010).

Integrability and Measurability

Theorem (García, Martín, Seoane, 2009)

On any interval I, the set of Lebesgue integrable functions that are not Riemann integrable is **spaceable***.*

Theorem (García, Martín, Seoane, 2009)

Given any unbounded interval I, the set of Riemann integrable functions on I that are not Lebesgue integrable is **lineable***.*

Theorem (Bernal, Ordóñez-Cabrera, 2013)

Given any unbounded interval I, the set of Riemann integrable functions on I that are not Lebesgue integrable is **spaceable***.*

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and denote by $N(\Omega, \mathbb{K})$ the set of non-measurable functions from Ω to $\mathbb{K}$, provided a measurable space (Ω, Σ).

Theorem (García, Seoane, 2006)

For any cardinal γ there is a Hausdorff topological space Ω with Borel σ-algebra $\mathcal{B}$ such that $N(\Omega, \mathbb{K}) \cup \{0\}$ contains a subspace isometric to $\ell_\infty(\gamma)$, i.e. $N(\Omega, \mathbb{K})$ is spaceable.
In particular, any Banach space with density character γ is isometric to a space consisting (but zero) of non–measurable functions.

The Denjoy-Clarkson property

It is well known that derivatives of functions of one real variable satisfy the Denjoy-Clarkson property:

> *If $u : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable, then the counterimage through u' of any open subset of $\mathbb{R}$ is either empty or has positive Lebesgue measure.*

Extending this result to several real variables is known as the *"Weil Gradient Problem"* and, after being an open problem for almost 40 years, was eventually solved in the negative for $\mathbb{R}^2$ by Buczolich in 2002.

Theorem (García, Grecu, Maestre, Seoane, 2010)

For every $n \geq 2$ there exists an infinite dimensional Banach space of differentiable functions on $\mathbb{R}^n$ which (except for 0) fail the Denjoy-Clarkson property.

Sierpiński-Zygmund functions

Theorem (Blumberg, 1922)

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. There exists a dense subset $S \subset \mathbb{R}$ such that the function $f|_S$ is continuous.

A careful reading of the proof of this result shows that the above set S is countable. Naturally, we could wonder whether we can choose the subset S in Blumberg's theorem to be **uncountable**. A (partial) negative answer to this was given by Sierpiński and Zygmund:

Theorem (Sierpiński, Zygmund, 1923)

There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that, for any set $Z \subset \mathbb{R}$ of cardinality the continuum, the restriction $f|_Z$ is not a Borel map.

$$\mathcal{SZ}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \ : \ f \text{ is a Sierpiński-Zygmund function} \}$$

Theorem (Gámez, Muñoz, Sánchez, Seoane, 2010)

$\mathcal{SZ}(\mathbb{R})$ is κ-lineable for some cardinal κ with $\mathfrak{c} < \kappa \leq 2^{\mathfrak{c}}$. Assuming the Generalized Continuum Hypothesis, $\mathcal{SZ}(\mathbb{R})$ is $2^{\mathfrak{c}}$-lineable.

Question

Can the $2^{\mathfrak{c}}$-lineability of $\mathcal{SZ}(\mathbb{R})$ be obtained in ZFC?

Theorem (Gámez, Seoane, 2013)

The $2^{\mathfrak{c}}$-lineability of $\mathcal{SZ}(\mathbb{R})$ in undecidable.

1-lineable and not 2-lineable sets, I

Example 1: Gurariy and Quarta (2005)

Let $\widehat{C}[0,1]$ be the subset of $C[0,1]$ of functions admitting one (and only one) absolute maximum. If $V \subset \widehat{C}[0,1] \cup \{0\}$ is a non-trivial linear space, then V is $1-$dimensional.

Recently, Botelho, Cariello, Fávaro, Pellegrino, and Seoane have obtained generalizations of the above result in a more general framework and for *bigger* dimensions.

1-lineable and not 2-lineable sets, II

Example 2: Albuquerque (2013)

Let us suppose that there exists a 2-dimensional vector space of injective functions, V, generated by f and g. Take $x \neq y$ and

$$\alpha = \frac{f(x) - f(y)}{g(y) - g(x)} \in \mathbb{R}.$$

Consider the function $h = f + \alpha g \in V \setminus \{0\}$.
By construction we have $h(x) = h(y)$.

EXAMPLE: non-lineable, n-lineable set ($\forall n \in \mathbb{N}$)

Let $j_1 \leq k_1 < j_2 \leq \cdots \leq k_m < j_{m+1} \leq \cdots$ integers. The set

$$M = \bigcup_m \left\{ \sum_{i=j_m}^{k_m} a_i x^i \ : \ a_i \in \mathbb{R} \right\}$$

is n-lineable for every $n \in \mathbb{N}$ and it is not lineable in $\mathcal{C}[0,1]$.

EXAMPLE: Totally non-linear sets

There are totally non-linear sets (but they might contain a positive cone).

EXAMPLE

Let X be an infinite dimensional Banach space. There exists a subset $M \subset X$ such that M is **spaceable and dense**, although it is **not dense-lineable**.

EXAMPLE

Every infinite dimensional Banach space X contains a subset M which is **lineable and dense**, but which is **not spaceable**. If X is separable, then M can also be chosen to be dense-lineable.

Lineability, Spaceability, ... What about other structures?

We have, so far, studied the existence of linear subspaces inside sets of functions enjoying some "*exotic*" property.

But, can we also construct algebras inside those sets of functions?

Aron, Pérez, and S. introduced the concept of **Algebrability** (originally coined by V. I. Gurariy) in

Studia Math. 175 (2006), 83–90,
and
Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 25–31.

Definition (Gurariy + Aron, Pérez, and Seoane, 2006)

- We say that a subset of functions M is **(μ, s)-algebrable** if $M \cup \{0\}$ contains an algebra $\mathcal{A}$ such that $\dim(\mathcal{A}) = \mu$ (as a vector space) and $\text{card}(S) = s$, where μ and s are two cardinal numbers and S is a minimal system of generators of $\mathcal{A}$.
- For short, M is said to be **algebrable** if $M \cup \{0\}$ contains an infinitely generated algebra.

ALGEBRABLE $\Rightarrow$ LINEABLE

Theorem (Aron, Pérez, Seoane, 2006)

*Let $E \subset \mathbb{T}$ be a set of measure zero. Let $\mathcal{F}_E \subset \mathcal{C}(\mathbb{T})$ be the set of continuous functions whose Fourier series expansion diverges at every point $t \in E$. Then $\mathcal{F}_E$ is (dense-)***algebrable***.*

Theorem (Aron, Conejero, Peris, Seoane, 2010)

There exists an **uncountably generated** *algebra every non-zero element of which is an everywhere surjective function on $\mathbb{C}$, that is, a function $f : \mathbb{C} \to \mathbb{C}$ such that, for every non void open set $U \subset \mathbb{C}$, $f(U) = \mathbb{C}$.*

Theorem (Gámez, Muñoz, Sánchez, Seoane, 2010)

$\mathcal{SZ}(\mathbb{R})$ *is* $(\mathfrak{c}, \mathfrak{c})$*-algebrable.*

Theorem (Rosenthal, 1968)

$\ell_\infty \setminus c_0$ *is spaceable.*

Theorem (García, Martín, Seoane, 2009)

$\ell_\infty \setminus c_0$ *is algebrable.*

Theorem (García, Martín, Seoane, 2009)

$\ell_\infty(\Gamma) \setminus c_0(\Gamma)$ *is spaceable and algebrable for every infinite set* Γ*.*

How "strange" can a function be?

A function from $ES(\mathbb{R})$ (I)

- Let $(I_n)_{n\in\mathbb{N}}$ be the collection of all open intervals with rational endpoints.
- I_1 contains a Cantor type set, call it C_1.
- $I_2 \setminus C_1$ also contains a Cantor type set, call it C_2.
- $I_3 \setminus (C_1 \cup C_2)$ contains, as well, a Cantor type set, C_3.
- Inductively... there exists a family of pairwise disjoint Cantor type sets, $(C_n)_{n\in\mathbb{N}}$, such that for every $n \in \mathbb{N}$,

$$I_n \setminus \left(\bigcup_{k=1}^{n-1} C_k \right) \supset C_n.$$

A function from $ES(\mathbb{R})$ (II)

- Next, take (for every $n \in \mathbb{N}$) ϕ_n any bijection

$$\phi_n : C_n \leftrightarrow \mathbb{R}.$$

- Define $f \in \mathbb{R}^{\mathbb{R}}$ as $f(x) = \begin{cases} \phi_n(x) & \text{if } x \in C_n, \\ 0 & \text{otherwise.} \end{cases}$
- f is clearly everywhere surjective (and also zero almost everywhere!).
- Indeed, let I be any interval in $\mathbb{R}$. There exists $k \in \mathbb{N}$ such that $I_k \subset I$, thus

$$f(I) \supset f(I_k) \supset f(C_k) = \phi_k(C_k) = \mathbb{R}.$$

∎

An even more "*pathological*" function

- ✓ **F. B. Jones** (1942) proved the existence of a function such that its graph intersects every closed subset of $\mathbb{R}^2$ with uncountable projection on the x-axis.
- ✓ A Jones function has dense graph in $\mathbb{R}^2$.
- ✓ If f is a Jones function, then $f(I) = \mathbb{R}$ for every interval I,
- ✓ also, f attains every real value "$\mathfrak{c}$ times", and
- ✓ moreover, $f(P) = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$.
- ✓ Jones functions are **not** measurable.

An even more "*pathological*" function

- ✓ **J. L. Gámez** (2011) proved that the set of *Jones functions* is actually $2^{\mathfrak{c}}$-lineable.
- ✓ **Ciesielski, Gámez, Pellegrino, Seoane** (2014) proved that the set of *Jones functions* is actually $2^{\mathfrak{c}}$-spaceable with respect to the topology of pointwise convergence.
- ✓ **Algebrability?**

Annulling functions in $\mathcal{C}[0,1]$

Definition

A function $f \in \mathcal{C}[0,1]$ is said to be an annulling function *if f has infinitely many zeros in $[0,1]$.*

It is easy to construct a $\mathfrak{c}-$generated algebra of annulling functions in $\mathcal{C}[0,1]$. But...

Is the set of annulling functions spaceable in $\mathcal{C}[0,1]$?

Annulling functions and spaceability

Answer:

Theorem (Enflo, Gurariy, Seoane, 2012)

Let X be any infinite dimensional closed subspace of $\mathcal{C}[0,1]$. There exists:

- *An infinite dimensional closed subspace Y of X, and*
- *a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0,1]$ (of pairwise different elements),*

such that $y(t_k) = 0$ for every $k \in \mathbb{N}$ and every $y \in Y$.

Related to the study of the amount of zeros of functions on a given interval, let us recall a question posed by Aron and Gurariy in 2003:

Is there an infinite dimensional subspace of ℓ_∞ every non-zero element of which has a finite number of zero coordinates?

Theorem (Cariello, Seoane, 2013)

Let X be c_0 or ℓ_p for $p \in [1, \infty]$. And denote by $Z(X)$ the subset of X formed by sequences having only a finite number of zero coordinates.

- $Z(X)$ does not contain infinite dimensional closed subspaces.
- $Z(X)$ is maximal algebrable and maximal lineable.

$Z(c_0)$ and $Z(\ell_p)$ are max. algebrable for $p \in [1, +\infty]$

For every real number $p \in]0, 1[$ denote

$$x_p = \left(p^1, p^2, p^3, \ldots\right),$$

and let $V = \text{span}\{x_p : p \in]0, 1[\}$. Notice that $V \subset X$, for $X = c_0$ or ℓ_p, $p \in [1, +\infty]$.
Next, take any $x \in V \setminus \{0\}$.We shall show that $x \in Z(X)$.We can write x as

$$x = \sum_{j=1}^{N} \lambda_j x_{p_j},$$

with $N \in \mathbb{N}$, $p_j \in]0, 1[$ for every $j \in \{1, 2, \ldots, N\}$, $p_N > p_{N-1} > \ldots > p_1$, and $(\lambda_j)_{j=1}^{N} \subset \mathbb{C}$.

Let us suppose that there exists an increasing sequence of positive integers $(m_k)_{k\in\mathbb{N}}$ such that $x(m_k) = 0$ for every $k \in \mathbb{N}$. Then, we have

$$0 = \sum_{j=1}^{N} \lambda_j p_j^{m_k}$$

for every $k \in \mathbb{N}$. Dividing the last identity by $p_N^{m_k}$, we obtain (for every $k \in \mathbb{N}$),

$$0 = \sum_{j=1}^{N-1} \lambda_j \left(\frac{p_j}{p_N}\right)^{m_k} + \lambda_N.$$

Now, since $0 < \frac{p_j}{p_N} < 1$ for every $j \in \{1, 2, \ldots, N-1\}$ and $\lim_{k\to\infty} m_k = \infty$, we have $\lim_{k\to\infty} \left(\frac{p_j}{p_N}\right)^{m_k} = 0$. Thus, $\lambda_N = 0$. By induction, we can easily obtain $\lambda_j = 0$ for every $j \in \{1, 2, \ldots, N\}$.This is a contradiction, since $x \neq 0$.

This argument also shows that the set

$$\{x_p : p \in]0, 1]\}$$

is linearly independent, thus V is $\mathfrak{c}$-dimensional (where $\mathfrak{c}$ stands for the continuum) and, thus, $Z(X)$ is maximal lineable for $X = c_0$ or ℓ_p, $p \in [1, +\infty]$.

Now let

$$x_p, x_q \in \{x_r,\ r \in \,]0,1[\ \}.$$

Notice that the coordinatewise product of x_p and x_q is

$$x_{pq} \in \{x_r,\ r \in \,]0,1[\ \}.$$

Therefore the algebra generated by $\{x_r,\ r \in \,]0,1[\ \}$ is the subspace generated by $\{x_r,\ r \in \,]0,1[\ \}$ which is V.
Consider any countable subset $W \subset V$. The subalgebra generated by W is a vector space generated by finite products of elements of W, but the set of finite products of elements belonging to a countable set is still countable. Therefore the subalgebra generated by W has countable dimension and, thus, W cannot be a set of generators for the algebra V, since $\dim(V)$ is uncountable. Therefore any set of generators of V is uncountable.

Peano maps

For any topological space X, let

$\mathcal{CS}_\infty(\mathbb{R}^m, X) := \left\{ f \in \mathcal{C}(\mathbb{R}^m, X) : f^{-1}(\{a\}) \text{ is unbounded for every } a \in X \right\}.$

In 2014 Albuquerque showed the following:

Theorem (Albuquerque, 2014)

For every pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}(\mathbb{R}^m, \mathbb{R}^n)$ is maximal lineable in the space $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

Also, more recently, Bernal and Ordóñez provided the following natural generalization of Albuquerque's result.

Theorem (Bernal and Ordóñez, 2014)

For each pair $m, n \in \mathbb{N}$, the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$ is maximal dense-lineable and spaceable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$. In particular, it is maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$.

A natural question would be to ask about the algebrability of the set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{R}^n)$.

Clearly, **algebrability** cannot be obtained in the real context, since for any $f \in \mathbb{R}^\mathbb{R}$, f^2 is always non-negative.

However, in the complex frame it is actually possible to obtain algebrability. Before that, let us recall some results related to **the growth** of an entire function.

Growth of an entire function

$\mathcal{H}(\mathbb{C})$ denotes the space of all entire functions from $\mathbb{C}$ to $\mathbb{C}$. For $r > 0$ and $f \in \mathcal{H}(\mathbb{C})$, we set $M(f, r) := \max_{|z|=r} |f(z)|$. $M(f, \cdot)$ increases strictly towards $+\infty$ as long as f is non-constant.

The (growth) *order* $\rho(f)$ of an entire function $f \in \mathcal{H}(\mathbb{C})$ is defined as the infimum of all positive real numbers α with the following property: $M(f, r) < e^{r^\alpha}$ for all $r > r(\alpha) > 0$. Note that $\rho(f) \in [0, +\infty]$. The order of a constant map is 0. If f is non-constant, we have

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log \log M(f, r)}{\log r}.$$

Some properties and examples, I

(a) If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is the MacLaurin series expansion of f then

$$\rho(f) = \limsup_{n\to+\infty} \frac{n \log n}{\log\left(1/|a_n|\right)}.$$

In particular, given $\alpha > 0$, $f_\alpha(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^{n/\alpha}}$ satisfies $\rho(f_\alpha) = \alpha$.

(b) For every $f \in \mathcal{H}(\mathbb{C})$, every $N \in \mathbb{N}$ and every $\alpha \in \mathbb{C} \setminus \{0\}$,

$$\rho\left(\alpha f^N\right) = \rho(f).$$

Some properties and examples, II

(c) For every $f, g \in \mathcal{H}(\mathbb{C})$,

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\}$$

and

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

Moreover, if f and g have different orders, then

$$\rho(f + g) = \max\{\rho(f), \rho(g)\} = \rho(f \cdot g),$$

where it is assumed $f \not\equiv 0 \not\equiv g$ for the second equality.

(d) *(Corollary to Hadamard's theorem)*: Every non-constant entire function f with $\infty > \rho(f) \notin \mathbb{N}$ is surjective.

An "interesting" consequence

Given any non-constant polynomial in M complex variables $P \in \mathbb{C}[z_1, \ldots, z_M]$, let $\mathcal{I}_P \subset \{1, \ldots, M\}$ be the set of indexes k such that the variable z_k explicitly appears in some monomial (with non-zero coefficient) of P; that is,

$$\mathcal{I}_P = \left\{ n \in \{1, \ldots, M\} : \frac{\partial P}{\partial z_n} \not\equiv 0 \right\}.$$

Proposition

Let $f_1, \ldots, f_M \in \mathcal{H}(\mathbb{C})$ such that $\rho(f_i) \neq \rho(f_j)$ whenever $i \neq j$. Then

$$\rho\left(P\left(f_1, \ldots, f_M\right)\right) = \max_{k \in \mathcal{I}_P} \rho\left(f_k\right),$$

Moreover, the set $(f_k)_{k=1}^{M}$ is algebraically independent.

The result (maximal algebrability)

Theorem (Albuquerque, Bernal, Pellegrino, Seoane, 2014)

The set $\mathcal{CS}_\infty(\mathbb{R}^m, \mathbb{C}^n)$ *is maximal algebrable in* $\mathcal{C}(\mathbb{R}^m, \mathbb{C}^n)$.

PROOF - 1/2

It suffices to consider the case $n = m = 1$.In fact, the case $m > 1$ follows from the $m = 1$ by considering the projection map from $\mathbb{R}^m$ to $\mathbb{R}$.The case $n > 1$ is obtained from $n = 1$ by working on each coordinate.

For each $s > 0$, select an entire function $\varphi_s : \mathbb{C} \to \mathbb{C}$ of order $s > 0$.Let $A := (0, +\infty) \setminus \mathbb{N}$.The Proposition assures that the set $\{\varphi_s\}_{s \in A}$ is a system of cardinality $\mathfrak{c}$ generating an algebra $\mathcal{A}$.

Next, notice that any element $\varphi \in \mathcal{A} \setminus \{0\}$ may be written as a non-constant polynomial P without constant term evaluated on some φ_{s_1}, φ_{s_2}, ..., φ_{s_N}:

$$\varphi = P(\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_N}) = \sum_{|\alpha| \leq m} c_\alpha \cdot \varphi_{s_1}^{\alpha_1} \cdot \varphi_{s_2}^{\alpha_2} \cdots \varphi_{s_N}^{\alpha_N}.$$

PROOF - 2/2

By the Proposition, there exists $j \in \{1, \dots, N\}$ such that

$$\rho(\varphi) = \rho(\varphi_{s_j}) = s_j \notin \mathbb{N}_0.$$

Thus, φ is surjective (Why?).
Finally, take any $F \in \mathcal{CS}_\infty(\mathbb{R}, \mathbb{C})$ and consider the algebra

$$\mathcal{B} := \{\varphi \circ F\}_{\varphi \in \mathcal{A}}.$$

Then it is plain that $\mathcal{B}$ is $\mathfrak{c}$-generated and that

$$\mathcal{B} \setminus \{0\} \subset \mathcal{CS}_\infty(\mathbb{R}, \mathbb{C}),$$

as required.

Peano and $\sigma-$Peano spaces

A theorem of Hahn and Mazurkiewicz provides a topological characterization of Hausdorff topological spaces that are continuous image of the unit interval $I := [0, 1]$: these are precisely the Peano spaces. We are interested in investigating the topological spaces that are continuous image of the *real line*,and for this task the following definition seems natural.

Definition

A topological space X is a *σ-Peano space* if there exists an increasing sequence of subsets

$$K_1 \subset K_2 \subset \cdots \subset K_m \subset \cdots \subset X,$$

such that each one of them is a Peano space (endowed with the topology inherited from X) and $\bigcup_{n\in\mathbb{N}} K_n = X$.

Proposition

Let X be a Hausdorff topological space. TFAE:

(a) X *is a σ-Peano space.*

(b) $\mathcal{CS}_\infty(\mathbb{R}, X) \neq \varnothing$.

(c) $\mathcal{CS}(\mathbb{R}, X) \neq \varnothing$.

Example (Spaces that are σ-Peano)

The Euclidean spaces $\mathbb{R}^n$ and Peano spaces are σ-Peano. For $1 < p \leq \infty$, the Hilbert cube

$$\mathrm{C}_p := \prod_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right] \subset \ell_p,$$

considered as a topological subspace of ℓ_p, is a compact metric space, so it is a Peano space. For each natural k, let $k\,\mathrm{C}_p$ be the Hilbert cube after applying an "k-homogeneous dilation" to it. Therefore, the union of Hilbert cubes $\bigcup_{k\in\mathbb{N}} k\,\mathrm{C}_p$ is a σ-Peano topological vector space, when endowed with the topology inherited from ℓ_p.

Example (Spaces that are NOT σ-Peano)

(a) Every σ-Peano space is separable (continuity preserves separability). In particular, ℓ_∞ is not σ-Peano.

(b) No infinite dimensional F-space is σ-compact (i.e., a countable union of compact spaces), and, therefore, is not σ-Peano. This is a consequence of the Baire category theorem combined with the fact that on infinite dimensional topological vector spaces, compact sets have empty interior. In particular, no infinite dimensional Banach space is σ-Peano.

Theorem (Albuquerque, Bernal, Pellegrino, Seoane, 2014)

Let $\mathcal{X}$ be a a σ-Peano topological vector space.
Then

$$\mathcal{CS}_\infty(\mathbb{R}^m, \mathcal{X})$$

is maximal lineable in $\mathcal{C}(\mathbb{R}^m, \mathcal{X})$.

Constructive techniques?

General (existence) results?

Definition (Aron, Gurariy, Seoane, 2004)

A subset M of a topological vector space X is said to be **dense-lineable** in X if there exists an infinite dimensional linear manifold in $M \cup \{0\}$ and dense in X.

Definition

Let A, B be subsets of a vector space X. We say that A is stronger than B if $A + B \subseteq A$.

In general, $0 \notin A$.
The following result will be give us a technique to prove the dense-lineability of certain lineable sets.

Theorem (Aron, García, Pérez, Seoane, 2009)

Let X be a separable Banach space, and consider two subsets A, B of X such that A is lineable and B dense-lineable.
If A is stronger than B, then A is dense-lineable.

Applications

$\mathcal{CND}[0,1]$

1. Gurariy (1966) showed that $\mathcal{CND}[0,1]$ is lineable.
2. The set $\mathcal{P}$ of polynomials in $[0,1]$ is dense in $\mathcal{C}[0,1]$.
3. Clearly, $\mathcal{CND}[0,1]$ is stronger than $\mathcal{P}$.
4. Thus, $\mathcal{CND}[0,1]$ is dense-lineable.

$\mathcal{HC}(D)$

1. In 1952, MacLane constructed a universal entire function for the differentiation operator

$$\begin{array}{rccc} D & : & \mathcal{H}(\mathbb{C}) & \longrightarrow & \mathcal{H}(\mathbb{C}) \\ & & f(z) & \mapsto & f'(z) \end{array}$$

2. Thus, there exists an entire function f such that the set $\{f^{(n)} : n \in \mathbb{N}\}$ is dense in $\mathcal{H}(\mathbb{C})$, endowed with the compact-open topology.
3. $HC(D)$ is stronger than $\mathcal{P}$.
4. Therefore, $HC(D)$ is dense-lineable.

$\mathcal{C}^\infty$ non-analytic functions

1. The set of non-analytic $\mathcal{C}^\infty$ functions on $[-1, 1]$, as well as the set of functions that are in $\mathcal{C}^m \setminus \mathcal{C}^n$ (with $m < n$), are dense-lineable.
2. Consider the infinite dimensional subspace S of $\mathcal{C}^\infty[-1, 1]$ given by $S = \text{span}\{e^{-\frac{\alpha}{x^2}} : \alpha > 0\}$. It is clear that every function in $S \setminus \{0\}$ is non-analytic, since all derivatives at 0 are equal to 0. However the set of $\mathcal{C}^\infty$ non-analytic functions is clearly stronger than the set of polynomials, which is dense.

Let us see that the set of $\mathcal{C}^\infty$-functions on $\mathbb{R}$ with constant Taylor expansion is maximal-algebrable.

$\mathcal{C}^\infty$ non-analytic functions, I

Let $M_0 \subset M$ be the set of $\mathcal{C}^\infty$-functions whose Taylor expansion is identically equal to zero. Consider the functions f_β, where $\beta \in \mathbb{R}^+$, given by

$$f_\beta(x) = \begin{cases} e^{-\beta x^{-2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

A straightforward calculation gives that $f_\beta^{(n)}(0) = 0$ for every $n \in \mathbb{N}$. Therefore $f_\beta \in M_0$.

Define now the following algebra:

$$A := \mathcal{A}\left(\{f_\gamma : \gamma \in \mathcal{H}\}\right).$$

II

The family $\{f_\gamma : \gamma \in \mathcal{H}\}$ is algebraically independent. Take any polynomial

$$P(f_{\gamma_1}, \ldots, f_{\gamma_k}) := \sum_{i=1}^{m} \alpha_i \prod_{j=1}^{k} f_{\gamma_j}^{n_{j,i}},$$

where $\{\alpha_i\}_{i=1}^m \subset \mathbb{R} \setminus \{0\}$, m, k and all the $n_{j,i}$'s are natural numbers and where $n_{j,a} = n_{j,b}$ for all $j \in \{1, \ldots, k\}$ if and only if $a = b$.

A straight forward calculation shows that

$$\begin{aligned} P(f_{\gamma_1}, \ldots, f_{\gamma_k}) &= \sum_{i=1}^{m} \alpha_i e^{-(n_{1,i}\gamma_1 + \ldots + n_{k,i}\gamma_k)x^{-2}} \\ &=: \sum_{i=1}^{m} \alpha_i e^{-\beta_i x^{-2}}. \end{aligned}$$

Since $\{\gamma_i\}_{i=1}^k \subset \mathcal{H}$, we get that all the β_i's are different positive real numbers.

III

Thus,

$$P(f_{\gamma_1}, \ldots, f_{\gamma_k})(x) = \sum_{i=1}^{m} \alpha_i f_{\beta_i}(x) =: F(x),$$

where $\beta_i = \beta_j$ if and only if $i = j$.
Suppose now that $F \equiv 0$. Then $F' \equiv 0$ and thus,

$$F'(x) = \frac{2}{x^3} \sum_{i=1}^{m} \alpha_i \beta_i f_{\beta_i}(x) =: \frac{2}{x^3} G_1(x) = 0 \quad \forall x \in \mathbb{R}.$$

Therefore $G_1 \equiv 0$ and so we must have $G_1' \equiv 0$, from which it follows that... $G_2 \equiv 0$... and so on.

IV

Continuing in the same manner we obtain that

$$G_n := \sum_{i=1}^{m} \alpha_i \beta_i^n f_{\beta_i} \equiv 0 \, \forall \in \mathbb{N} \Rightarrow 0 = \lim_{x\to\infty} G_n(x) = \sum_{i=1}^{m} \alpha_i \beta_i^n \, \forall \in \mathbb{N}$$

By writing this last equation for $n = 0, \ldots, m-1$, we obtain:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_m \\ \beta_1^2 & \beta_2^2 & \beta_3^2 & \cdots & \beta_m^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_1^{m-1} & \beta_2^{m-1} & \beta_3^{m-1} & \cdots & \beta_m^{m-1} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

A Vandermonde-type matrix!!!!

Therefore, $\alpha_1 = \ldots = \alpha_k = 0 \Longrightarrow \{f_\gamma : \gamma \in \mathcal{H}\}$ is a. i.

Questions

- Is there any general (existence, non-constructive) result that guarantees the lineability of a given subset of $\mathbb{R}^{\mathbb{R}}$?

- On the other hand... Is there an "easy" way to find out when a set is not lineable at all?

Additivity

- We have given conditions that provide dense lineability for lineable sets.
- However, Is it possible to know when a subset of $\mathbb{R}^{\mathbb{R}}$ is, simply, **lineable**?
- Let us recall the following concept:

Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. The **additivity** of $\mathcal{F}$ is given by the following cardinal invariant:

$$\mathcal{A}(\mathcal{F}) = \min\big(\{\,\text{card}\, F : F \subset \mathbb{R}^{\mathbb{R}},\ \varphi + F \not\subset \mathcal{F}, \forall \varphi \in \mathbb{R}^{\mathbb{R}}\,\} \cup \{(2^{\mathfrak{c}})^{+}\}\big),$$

where $(2^{\mathfrak{c}})^{+}$ denotes the successor of $2^{\mathfrak{c}}$.

$$\mathcal{A}(\mathcal{F}) = \min\left(\{\operatorname{card} F : F \subset \mathbb{R}^{\mathbb{R}},\ \varphi + F \not\subset \mathcal{F},\ \forall \varphi \in \mathbb{R}^{\mathbb{R}}\} \cup \{(2^{\mathfrak{c}})^{+}\}\right)$$

Additivity:

- Introduced by T. Natkaniec in the 1990's and thoroughly studied in F. E. Jordan's dissertation (1998).
- The additivity of an immense amount of families of functions is known.
- While knowing whether a set is lineable or not can be a very hard problem, finding the additivity of a certain set is (in general) fairly simple.

Additivity and lineability

Theorem (Gámez, Muñoz, Seoane, 2010)

Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ star-like. If $\mathfrak{c} < \mathcal{A}(\mathcal{F}) \leq 2^{\mathfrak{c}}$, then $\mathcal{F}$ is $\mathcal{A}(\mathcal{F})$-lineable.

This previous result allows us to obtain the lineability of many families of *strange* functions.

Some Definitions in Real Analysis

Let $f \in \mathbb{R}^{\mathbb{R}}$. We say that:

1.- $f \in \mathrm{ES}(\mathbb{R})$ (f is everywhere surjective) if $f(I) = \mathbb{R}$ for every non-trivial interval I.

2.- $f \in \mathrm{SES}(\mathbb{R})$ (f is strongly everywhere surjective) if f takes all values $\mathfrak{c}$ times on any interval.

3.- $f \in \mathrm{PES}(\mathbb{R})$ (f is perfectly everywhere surjective) if for every perfect set P, $f(P) = \mathbb{R}$.

4.- $f \in \mathrm{AC}(\mathbb{R})$ (f is almost continuous, in the sense of J. Stallings) if every open set containing the graph of f contains also the graph of some continuous function.

5.- If $h: X \to \mathbb{R}$, where X is a topological space, $h \in \mathrm{Conn}(X)$ (h is a connectivity function) if the graph of $h|_C$ is connected for every connected set $C \subset X$. (If $h \in \mathbb{R}^{\mathbb{R}}$, it is equivalent to say that its graph is connected.)

6.- $f \in \mathrm{Ext}(\mathbb{R})$ (f is extendable) if there is a connectivity function $g\colon \mathbb{R}^2 \to [0,1]$ such that $f(x) = g(x,0)$ for every $x \in \mathbb{R}$.

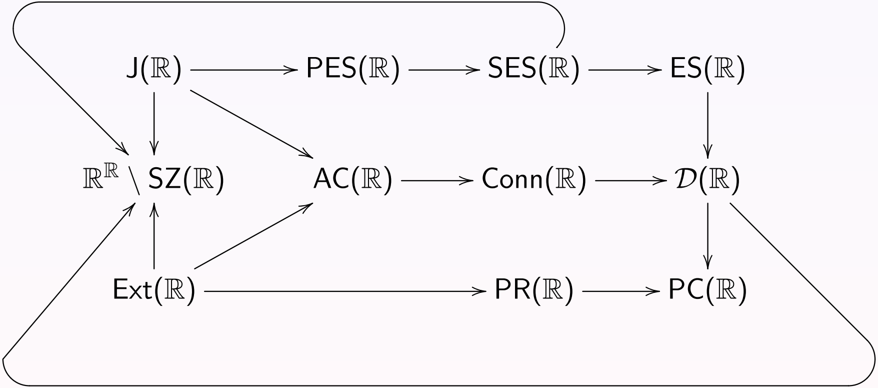
7.- $f \in \mathrm{PR}(\mathbb{R})$ (f is a perfect road function) if for every $x \in \mathbb{R}$ there is a perfect set $P \subset R$ such that x is a bilateral limit point of P and $f|_P$ is continuous at x.

8.- $f \in \mathrm{PC}(\mathbb{R})$ (f is peripherally continuous) if for every $x \in \mathbb{R}$ and pair of open sets $U, V \subset \mathbb{R}$ such that $x \in U$ and $f(x) \in V$ there is an open neighborhood W of x with $\overline{W} \subset U$ and $f(\mathrm{bd}(W)) \subset V$.

9.- $f \in \mathrm{SZ}(\mathbb{R})$ (f is a Sierpiński-Zygmund function) if f is not continuous on any set of cardinality $\mathfrak{c}$.

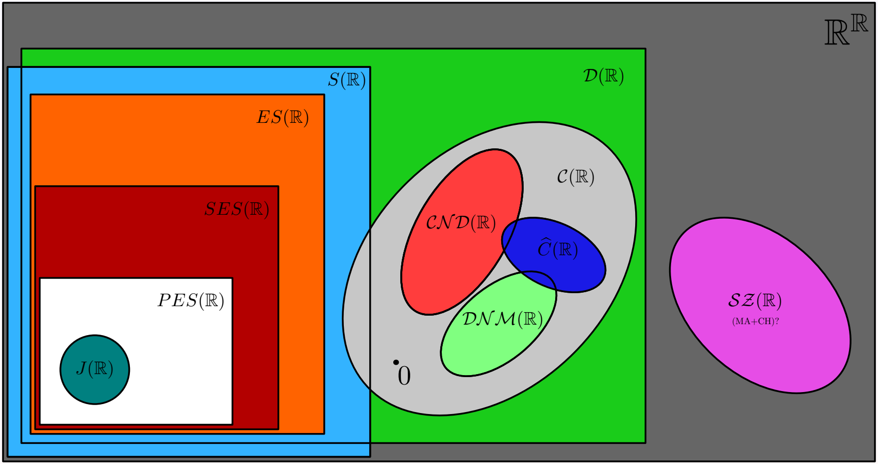
10.- $f \in \mathcal{D}(\mathbb{R})$ if f is a Darboux function.

Some relations between the previous concepts.



Sacks′ model/Iterated Perfect Set Model

Some relations between other subsets of $\mathbb{R}^{\mathbb{R}}$.



Lineability VIA additivity

Here we see that combining the additivity of a certain family plus the GCH we can obtain **sharp** results:

SET	ADDITIVITY	LINEABILITY (ZFC)	+ GCH
$AC(\mathbb{R})$	$e_{\mathfrak{c}}$	$2^{\mathfrak{c}}$	$=$
$Conn(\mathbb{R})$	$e_{\mathfrak{c}}$	$2^{\mathfrak{c}}$	$=$
$Ext(\mathbb{R})$	$\mathfrak{c}^+$	$2^{\mathfrak{c}}$	$=$
$PR(\mathbb{R})$	$\mathfrak{c}^+$	$2^{\mathfrak{c}}$	$=$
$SZ(\mathbb{R})$	$d_{\mathfrak{c}}$	$\mathfrak{c}^+$	$2^{\mathfrak{c}}$
$J(\mathbb{R})$	$e_{\mathfrak{c}}$	$2^{\mathfrak{c}}$	$=$

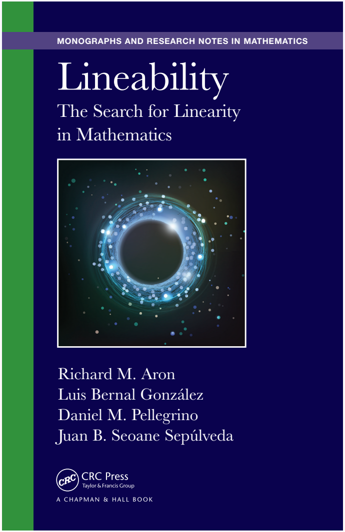
$$d_{\mathfrak{c}} = \min\{ \operatorname{card} F \ : \ F \subset \mathbb{R}^{\mathbb{R}}, (\forall \varphi \in \mathbb{R}^{\mathbb{R}})(\exists f \in F)(\operatorname{card}(f \cap \varphi) = \mathfrak{c}) \}$$
$$e_{\mathfrak{c}} = \min\{ \operatorname{card} F \ : \ F \subset \mathbb{R}^{\mathbb{R}}, (\forall \varphi \in \mathbb{R}^{\mathbb{R}})(\exists f \in F)(\operatorname{card}(f \cap \varphi) < \mathfrak{c}) \}$$

$$\mathfrak{c}^+ \leq d_{\mathfrak{c}} \leq 2^{\mathfrak{c}} \qquad \mathfrak{c}^+ \leq e_{\mathfrak{c}} \leq 2^{\mathfrak{c}}$$

General "existence" techniques

- **2011** (Kitson and Timoney).
- **2012** (Botelho, Cariello, Fávaro, and Pellegrino): Maximal-spaceability (applied to subsets of certain topological vector sequence spaces).
- **2013** (Bernal, Ordóñez): General technique that can be applied to many frameworks of study, generalizing one of the previous known ones.

some comercials!!!



THANK YOU

FOR YOUR ATTENTION!!!