Spaceability and operator ideals

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Definition (Aron, Gurariy, Seoane, 2005)

Given a topological vector space X, a subset $A \subset X$ is said to be **lineable** if $A \cup \{0\}$ contains an infinite-dimensional linear subspace.

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If E and F are Banach spaces where E has the two series property, then $L(E, F^*) \setminus \Pi_1(E, F^*)$ is lineable.

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If E is a superreflexive Banach space containing a complemented infinite-dimensional subspace with unconditional basis, or F is a Banach space having an infinite unconditional basic sequence, then $K(E, F) \setminus \Pi_p(E, F)$ is lineable for every $p \geq 1$.

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Theorem (Kitson, Timoney, 2011)

If E is a superreflexive Banach space, then $K(E, F) \setminus \bigcup_{p \geq 1} \Pi_p(E, F)$ is spaceable.

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- Is $I_1(E, F) \setminus I_2(E, F)$ spaceable?
- If I_1 and I_2 are Banach operator ideals such that $I_2 \subset I_1$ continuously and I_2 is not closed in I_1, then $I_1(E, F) \setminus I_2(E, F)$ is spaceable.

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- For each pair of Banach spaces E and F, $I(E, F)$ (or $I(E)$ if $E = F$) is a subspace of the space $L(E, F)$ (or $L(E)$ if $E = F$) of bounded linear operators from E to F containing all finite-rank operators.

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- If in a scheme of bounded linear operators $E_0 \xrightarrow{S_1} E \xrightarrow{T} F \xrightarrow{S_2} F_0$ we have $T \in I(E, F)$, then $S_2 \circ T \circ S_1 \in I(E_0, F_0)$.

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- $\|S_2 \circ T \circ S_1\|_I \leq \|S_2\|\|T\|_I\|S_1\|$ for $S_2 \in L(F, F_0)$, $T \in I(E, F)$ and $S_1 \in L(E_0, E)$.

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- An ideal norm is a norm.
- The usual operator norm of $L(E, F)$ is an ideal norm.
- $\|T\| \leq \|T\|_I$ for $T \in I$.

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- If E and F are σ-reproducible Banach spaces, then $E \oplus F$ and E^* are also σ-reproducible.

Rearrangement invariant spaces

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Definition

Given a measure space (Ω, λ), where $\Omega = [0,1], [0,\infty)$ and λ is the Lebesgue measure, or $\Omega = \mathbb{N}$ and λ is the counting measure, the **distribution function** λ_x associated to a scalar measurable function x on Ω is defined by $\lambda_x(s) = \lambda\{t \in \Omega : |x(t)| > s\}$.

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Proof

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First, let E be a rearrangement invariant space on $[0,1]$. For every $a \in [0,1)$ and $r \in (0, 1-a]$ we consider the complemented subspace

$$E_{a,r} = \{x \in E : \text{supp } x \subseteq [a, a+r]\}$$

and the bounded projection $P_{a,r} : E \longrightarrow E_{a,r}$ given by $P_{a,r}(x) = x\chi_{[a,a+r]}$ for $x \in E$.

Rearrangement invariant spaces

Proof (cont.)

For a measurable function x we define the linear operators

$$T_{a,r}(x)(t) = x\left(\frac{t-a}{r}\right)\chi_{(a,a+r]}(t)$$

$$S_{a,r}(x)(t) = x((1-t)a + t(a+r))$$

which are bounded from L^∞ to L^∞, from L^1 to L^1 and, then, from E to E (Calderón-Mitjagin interpolation theorem).

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which are bounded from L^∞ to L^∞, from L^1 to L^1 and, then, from E to E (Calderón-Mitjagin interpolation theorem). $(S_{a,r} \circ T_{a,r})(x) = x$ for every $x \in E$, $(T_{a,r} \circ S_{a,r})(x) = x$ for every $x \in E_{a,r}$ and $T_{a,r} : E \longrightarrow E_{a,r}$ and $S_{a,r} : E_{a,r} \longrightarrow E$ are isomorphisms.

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which are bounded from L^∞ to L^∞, from L^1 to L^1 and, then, from E to E (Calderón-Mitjagin interpolation theorem). $(S_{a,r} \circ T_{a,r})(x) = x$ for every $x \in E$, $(T_{a,r} \circ S_{a,r})(x) = x$ for every $x \in E_{a,r}$ and $T_{a,r} : E \longrightarrow E_{a,r}$ and $S_{a,r} : E_{a,r} \longrightarrow E$ are isomorphisms. For every $n \in \mathbb{N}$ we consider $a_n = 1 - \frac{1}{2^{n-1}}$ and $r_n = \frac{1}{2^n}$. Let $E_n = E_{a_n,r_n}$ and $P_n = P_{a_n,r_n}$. Since $\left\|\widetilde{P_k}\right\| = 1$ for all $k \in \mathbb{N}$, E is σ-reproducible.

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$$T_n(x)(t) = \sum_{k=1}^{\infty} x\left(t+k-1-a_{n,k}\right)\chi_{(a_{n,k},a_{n,k}+1]}(t)$$

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$(T_n(x))^* = x^*$, $(S_n(x))^* \leq x^*$ and $(S_n \circ T_n)(x) = x$. Then, $T_n : E \longrightarrow E_n$ is an isometry and $S_n : E_n \longrightarrow E$ is an isomorphism. Follow the $[0,1]$ case.

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$$a_n = \begin{cases} x_m & \text{if } \varphi_k(m) = n \\ 0 & \text{if } n \notin A_k \end{cases}$$

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$S_k(x) = (x_{\varphi_k(n)})_{n\in\mathbb{N}}$. If $E_k = \{x \in E : \operatorname{supp} x \subseteq A_k\}$, then $T_k : E \longrightarrow E_k$ is an isometry and $S_k : E_k \longrightarrow E$ is an isomorphism. And reasoning again as in the $[0,1]$ case we obtain the result.

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Proof

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$$P_{[a,b]}(f)(x) = \left(f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) \right) \chi_{[a,b]}(x).$$

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For an increasing sequence $(a_n)_{n\in\mathbb{N}} \subset (0,1)$, let $E_n = \widehat{C}[a_n, a_{n+1}]$.

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Let $T \in I_1(E,F) \setminus I_2(E,F)$.

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Let $T \in I_1(E,F) \setminus I_2(E,F)$. If E is a σ-reproducible Banach space with isomorphisms $\phi_n : E_n \longrightarrow E$ and bounded projections $P_n : E \longrightarrow E_n$, for every $n \in \mathbb{N}$ we consider the operator $T_n = T \circ \phi_n \circ P_n$ which belongs to $I_1(E,F) \setminus I_2(E,F)$. If $T_n \in I_2(E,F)$, then $T_n|_{E_n} = T \circ \phi_n \in I_2(E_n,F)$.

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Proof (cont.)

Furthermore, $(T_n)_{n\in\mathbb{N}}$ is a basic sequence in $I_1(E,F)$. Indeed, for any integers $k<m$ and any choice of scalars $(\lambda_n)_{n\in\mathbb{N}}$ we have

$$\left\|\sum_{n=1}^{k}\lambda_n T_n\right\|_{I_1}=\left\|\sum_{n=1}^{m}\lambda_n T_n\circ\widetilde{P_k}\right\|_{I_1}\leq\left\|\sum_{n=1}^{m}\lambda_n T_n\right\|_{I_1}\left\|\widetilde{P_k}\right\|.$$

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Let $S\in\overline{[T_n:n\in\mathbb{N}]}\subset I_1(E,F)$ with $S=\sum_{n=1}^{\infty}\lambda_n T_n\neq 0$. Then there exists $n_0\in\mathbb{N}$ such that $\lambda_{n_0}\neq 0$. We have that $S|_{E_{n_0}}=\lambda_{n_0}T\circ\phi_{n_0}\notin I_2(E_{n_0},F)$. Thus, $S\notin I_2(E,F)$ and $\overline{[T_n:n\in\mathbb{N}]}\subset I_1(E,F)\setminus I_2(E,F)$.

Proof (cont.)

If F is σ-reproducible with isomorphisms $(\phi_n)_{n\in\mathbb{N}}$, for each $n \in \mathbb{N}$ we consider the operator $T_n = \phi_n^{-1} \circ T$ which belongs to $I_1(E,F) \setminus I_2(E,F)$.

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$$\left\| \sum_{n=1}^{k} \lambda_n T_n \right\|_{I_1} = \left\| \widetilde{P_k} \circ \sum_{n=1}^{m} \lambda_n T_n \right\|_{I_1} \le \left\| \widetilde{P_k} \right\| \left\| \sum_{n=1}^{m} \lambda_n T_n \right\|_{I_1} .$$

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Let $S \in \overline{[T_n : n \in \mathbb{N}]} \subset I_1(E,F)$, $S = \sum_{n=1}^{\infty} \lambda_n T_n \neq 0$. There exists $n_0 \in \mathbb{N}$ such that $\lambda_{n_0} \neq 0$. If $S \in I_2(E,F)$, then $P_{n_0} \circ S \in I_2(E,F)$, but this is not true because $P_{n_0} \circ S = \lambda_{n_0} T_{n_0}$. Then $\overline{[T_n : n \in \mathbb{N}]} \subset I_1(E,F) \setminus I_2(E,F)$.

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If E or F is a σ-reproducible Banach space, I is an operator ideal such that $I(E,F)$ is complete for an ideal norm, and $(I_n)_{n\in\mathbb{N}}$ is a sequence of operator ideals such that $I(E,F)\setminus I_n(E,F)$ is non-empty for every $n\in\mathbb{N}$, then the set $I(E,F)\setminus\bigcup_{n=1}^{\infty} I_n(E,F)$ is spaceable.

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Proof

Let $S_n\in I(E,F)\setminus I_n(E,F)$ for every $n\in\mathbb{N}$. If E is a σ-reproducible Banach space with isomorphisms $(\phi_n)_{n\in\mathbb{N}}$ and bounded projections $(P_n)_{n\in\mathbb{N}}$, let us consider the operators $S_n\circ\phi_n\circ P_n\in I(E,F)\setminus I_n(E,F)$ for every $n\in\mathbb{N}$.

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$$T=\sum_{n=1}^{\infty}\frac{S_n\circ\phi_n\circ P_n}{2^n\|S_n\circ\phi_n\circ P_n\|_I}\in I(E,F)\setminus I_n(E,F).$$

Consequences

Proof (cont.)

Now, reasoning as in the proof of the main theorem we can construct a sequence $(T_k)_{k\in\mathbb{N}}$ such that $\overline{[T_k : k \in \mathbb{N}]} \subset I(E,F) \setminus I_n(E,F)$ for every $n \in \mathbb{N}$.

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$$T = \sum_{n=1}^{\infty} \frac{\phi_n^{-1} \circ S_n}{2^n \|\phi_n^{-1} \circ S_n\|_I}$$

belongs to $I(E,F) \setminus \bigcup_{n=1}^{\infty} I_n(E,F)$.

Consequences

Corollary

Let E and F be Banach spaces, and $\{I_p : p \in [a, b]\}$ be a family of operator ideals such that $I_p(E, F) \subsetneq I_q(E, F)$ if $p < q$ with continuous inclusion.

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Let E and F be Banach spaces, and $\{I_p : p \in [a, b]\}$ be a family of operator ideals such that $I_p(E,F) \subsetneq I_q(E,F)$ if $p < q$ with continuous inclusion. If E or F is a σ-reproducible Banach space and $I_b(E,F)$ is complete for an ideal norm, then the set $I_b(E,F) \setminus \bigcup_{p<b} I_p(E,F)$ is spaceable.

Applications: strictly singular operators

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- The set $SS(E, c_0) \setminus K(E, c_0)$ is spaceable for every symmetric sequence space $E \neq c_0$.

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If E is a σ-reproducible Banach space, then the set $SS(E) \setminus K(E)$ is spaceable if and only if E does not have the Kato property.

- For Lorentz function spaces $L^{p,q}[0,1]$ with $1 < p < \infty, 1 \leq q \leq \infty$, the set $SS(L^{p,q}) \setminus K(L^{p,q})$ is spaceable if and only if $q \neq 2$.

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- For 2-convex (or 2-concave) Orlicz spaces $L^{\varphi}[0,1]$ it holds that $SS(L^{\varphi}) \setminus K(L^{\varphi})$ is spaceable if and only if the associated set $E_{\varphi}^{\infty} \neq \{t^2\}$.

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A linear operator T between two Banach spaces E and F is called **finitely strictly singular** (FSS) if there do not exist a number $c > 0$ and a sequence of subspaces E_n of E with $\dim(E_n) = n$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in \bigcup_{n=1}^{\infty} E_n$.

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- $K \subset FSS \subset SS$.
- If $1 < p < q < \infty$, the sets $SS(\ell_p, \ell_q) \setminus FSS(\ell_p, \ell_q)$ and $FSS(\ell_p, \ell_q) \setminus K(\ell_p, \ell_q)$ are spaceable.

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- For the disc algebra $A(D)$, the set $FSS(A(D)) \setminus K(A(D))$ is spaceable but the set $SS(A(D)) \setminus FSS(A(D))$ is not spaceable.

Applications: disjointly strictly singular operators

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A linear operator T from a Banach lattice E to a Banach space F is said to be **disjointly strictly singular** (DSS) if there is no disjoint sequence of non-null vectors in E such that the restriction of T to the closed subspace spanned by them is an isomorphism.

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- For $L^p[0,1]$-spaces, $DSS(L^p) \setminus SS(L^p)$ is spaceable if $1 < p \neq 2$ (the projection of L^p over the closed subspace spanned by Rademacher functions is DSS but not SS).

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- $DSS(L^q, L^p) \setminus SS(L^q, L^p)$ is spaceable if $1 \leq p < q < \infty$ (the inclusion $L^q \hookrightarrow L^p$ is DSS but not SS).

Applications: (q, p)-summing operators

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$$\left(\sum_{i=1}^n \|T(x_i)\|^q\right)^{1/q} \leq C \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^n |x^*(x_i)|^p\right)^{1/p}.$$

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- For $1 \leq p \leq r \leq q$ it holds $\Pi_{q,q} = \Pi_q \subset \Pi_{q,r} \subset \Pi_{q,p} \subset \Pi_{q,1}$.

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- For $1 \leq p \leq r \leq q$ it holds $\Pi_{q,q} = \Pi_q \subset \Pi_{q,r} \subset \Pi_{q,p} \subset \Pi_{q,1}$.
- If H is a Hilbert space, then the set $\Pi_{q,1}(H) \setminus \bigcup_{1<p\leq q} \Pi_{q,p}(H)$ is spaceable if and only if $1 < q < 2$.