Genericity in (sub)spaces of continuous functions

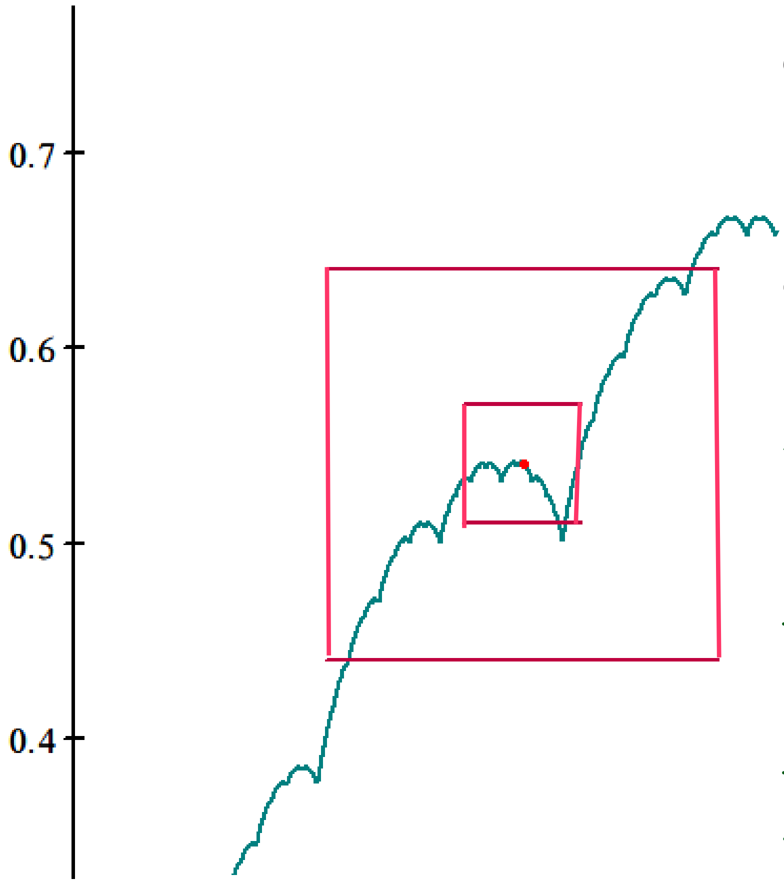
Zoltán Buczolich

Eötvös University, Budapest

www.cs.elte.hu/∼buczo

Micro tangent sets of typical/generic continuous functions

In Mathematical Reviews 97j:28009, the reviewer (Joan Verdera), wrote the following:
"Tangent measures play, with respect to measures, the same role that derivatives play with respect to functions.
Given a measure μ (locally finite Borel measure on $\mathbb{R}^n$) and a point, one looks at the measure in a small neighborhood of the point, blows it up, normalizes suitably and takes a weak star limit in the space of measures.
The result is a tangent measure for μ at the given point."
With the concept of micro tangent sets from measures we return to continuous functions and we see that this concept of blowing up and taking suitable limits, this time in the Hausdorff metric, might be useful to obtain information about "tangential regularity" of irregular functions.



The closed cube of side length $2\delta > 0$ centered at $(x;y)$ will be denoted by $Q((x;y),\delta)$, that is, $Q((x;y),\delta) = \{(x';y') : |x'-x| \le \delta \text{ and} |y'-y| \le \delta\}$.

Let Q^2 be the closed cube of side length 2, centered at $(0;0)$, that is, $Q((0;0),1)$.

D.:For $\delta_n > 0$ we put

$$F(f,x_0,\delta_n) = \frac{1}{\delta_n}\Big(\big(graph(f) \cap Q((x_0;f(x_0)),\delta_n)\big) - (x_0;f(x_0))\Big),$$

that is, $F(f,x_0,\delta_n)$ is the $1/\delta_n$-times enlarged part of $graph(f)$ belonging to $Q((x_0;f(x_0)),\delta_n)$ translated into Q^2.

The set F is a *micro tangent set (M-tangent set)* of f at x_0, that is, $F \in f_{MT}(x_0)$ if there exists $\delta_n \searrow 0$ such that $F(f,x_0,\delta_n)$ converges to F in the Hausdorff metric.

If f is differentiable at x_0 then $f_{MT}(x_0)$ consists of one line segment of slope $f'(x_0)$ passing through the origin.

By $C[-1,1]_0$ we mean the set of those functions g in $C[-1,1]$ for which $g(0)=0$.

D.:We say that x_0 is a *universal MT-point* for f if $graph(g) \cap Q^2 \in f_{MT}(x_0)$ for every $g \in C[-1,1]_0$.
The collection of those points $(x_0; f(x_0))$ for which x_0 is a universal MT-point of f will be denoted by $UMT(f)$.

A property is *typical/generic* if functions in $C[0,1]$ **not having** this property are of first Baire category.

T.:For any function $f \in C[0,1]$ the set $UMT(f)$ is of σ-finite $\mathcal{H}^1$-measure.

By a result of R. D. Mauldin and S. C. Williams the graph of the typical continuous function is of Hausdorff dimension one, but is not of σ-finite $\mathcal{H}^1$-measure.
So, the Theorem above says that most points in the sense of $\mathcal{H}^1$-measure on the graph of the typical continuous function are not universal.

T.: *There is a dense G_δ set $\mathcal{G}$ of $C[0,1]$ such that $\lambda(\pi_x(UMT(f))) = 1$ for all $f \in \mathcal{G}$. Furthermore, $UMT(f)$ is a dense G_δ subset in the relative topology of $graph(f)$. Hence, for the typical continuous function in $C[0,1]$ almost every $x \in [0,1]$ is a universal MT-point and a typical point on the graph of f is in $UMT(f)$.*

The next theorem shows that though $UMT(f)$ has large x-projection,
it has small y-projection.
T.: *There is a dense G_δ set $\mathcal{G}$ of $C[0,1]$ such that $\lambda(\pi_y(UMT(f))) = 0$ for all $f \in \mathcal{G}$. Hence any preimage of almost every y in the range of the typical continuous function is not a UMT-point.*

By considering functions $g_c(x) = f(x)+cx$, one can see that $UMT(f)$ projects to a set of λ-measure zero in any direction
different from the y-axis.This implies that $UMT(f)$ is a quite naturally defined irregular 1-set on the graph of a typical continuous function.
In a joint work with former Ph.D. student Cs. Ráti we considered the question:
What happens at other points of the typical continuous function?

Hölder exponent and singularity spectrum for a locally bounded function.

D.: Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $x \in [0,1]^d$, the function f belongs to C^h_x if there are a polynomial P of degree less than $[h]$ and a constant C such that, for x' close to x,

$$|f(x') - P(x'-x)| \leq C|x'-x|^h.$$

The pointwise Hölder exponent of f at x is $h_f(x) = \sup\{h \geq 0 : \ f \in C^h_x\}$.

When $h_f(x) < 1$, the pointwise Hölder exponent of f at x is also given by the formula

$$h_f(x) = \liminf_{x' \to x} \frac{\log |f(x') - f(x)|}{\log |x'-x|}.$$

D.: singularity spectrum of f is defined by

$$d_f(h) = \dim_H E^h_f, \quad \text{where} E^h_f = \{x : h_f(x) = h\}.$$

$\dim_H$ = Hausdorff dimension, and $\dim \emptyset = -\infty$ by convention.

We will also use the sets $E^{h,\leq}_f = \{x : h_f(x) \leq h\} \supset E^h_f$.

Hölder spectrum of monotone continuous functions

Let d be an integer greater than one.
A function $f : [0,1]^d \to \mathbb{R}$ is continuous monotone increasing in several variables (in short: MISV) if for all $i \in \{1, ..., d\}$,
the functions $f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)$
are continuous monotone increasing.

We set $\mathcal{M}^d = \{f \in C([0,1]^d) : f \text{ is MISV}\}$.

The space $\mathcal{M}^d$ is a separable complete metric space when equipped with the supremum norm for functions, that we denote by $\|.\|$.

The multifractal properties of functions in $\mathcal{M}^1$ have been examined by Z. B. and J. Nagy while the higher $n \geq 2$ dimensional cases were studied by Z. B. and S. Seuret.

Zamfirescu: the typical monotone continuous function, f is a strictly monotone increasing singular function and its derivative equals 0 wherever it exists. Of course, f' exists almost everywhere in $[0,1]$.

S. Jaffard has studied the multifractal properties of several specific continuous functions. For these functions many interesting methods, for example, wavelets, Diophantine approximation etc. were used.

It is not difficult to verify that the typical continuous function on $[0,1]$ does not have multifractal Hölder properties, (it is monofractal) in fact, it is Hölder class 0 everywhere in $[0,1]$. While the class of typical monotone continuous functions are of multifractal nature. By studying typical monotone continuous functions we study typical continuous measures on $[0,1]$.
Multifractal properties of generic measures on $[0,1]^d$ were studied by Z.B. and S. Seuret and on compact subsets of $\mathbb{R}^d$ by F. Bayart.

The typical (generic) properties of functions in $\mathcal{M}^1$:

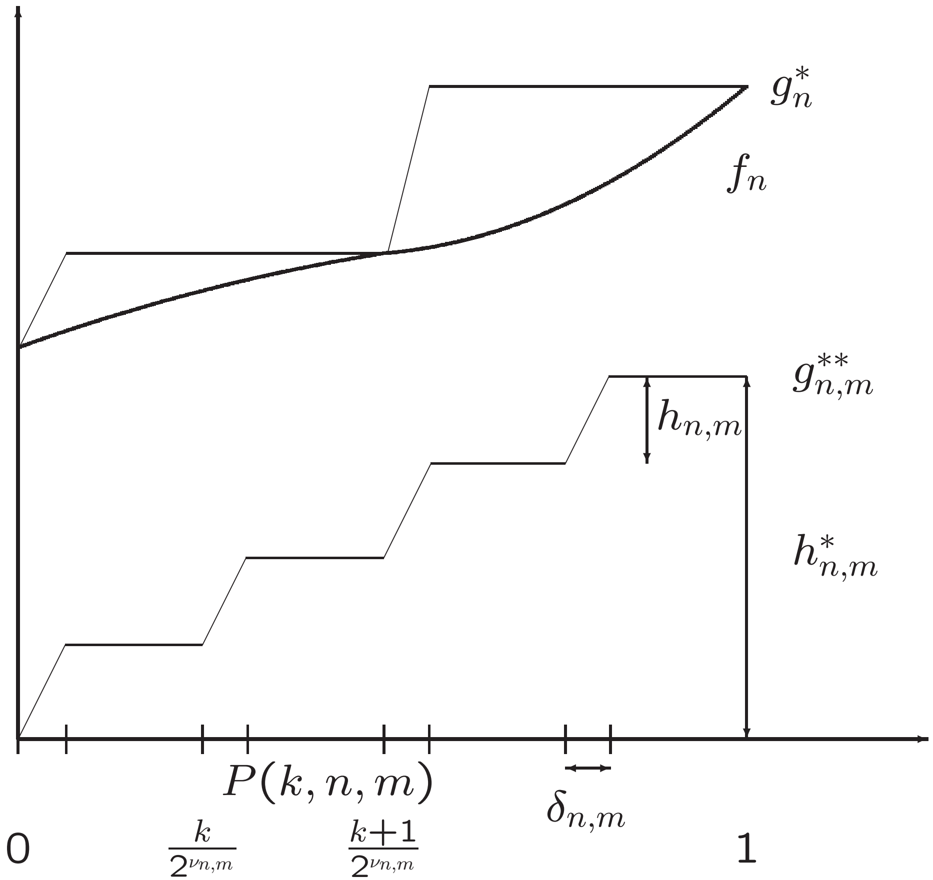
T.: (Z.B. & J. Nagy) *Consider the space of monotone continuous functions $\mathcal{M}^1$ defined on* $[0,1]$.

(i) For every $f \in \mathcal{M}^1$, for every $h \geq 0$, one has $d_f(h) \leq \min(h,1)$.

(ii) There exists a residual set $\mathcal{R}_1$ in $\mathcal{M}^1$ such that for every $f \in \mathcal{R}_1$, $d_f(h) = h$ for every $h \in [0,1]$, and $E_f^h = \emptyset$ if $h > 1$.

(iii) $\mu_f([0,1] \setminus E_f^{0,\leq}) = 0$, where μ_f is the Borel integral of f: $f(x) = \int_0^x d\mu_f$.

By (i) of the above theorem, $0 = \dim_H E_f^0 = \dim_H E_f^{0,\leq}$.
(iii) shows that all the "increasing" of f takes place on this set E_f^0 of zero Hausdorff dimension.
Since a typical monotone function is strictly monotone increasing, its level sets are points.
We deduce that heuristically, for "most" levels in the range of f, the corresponding points belong to the zero dimensional set E_f^0.

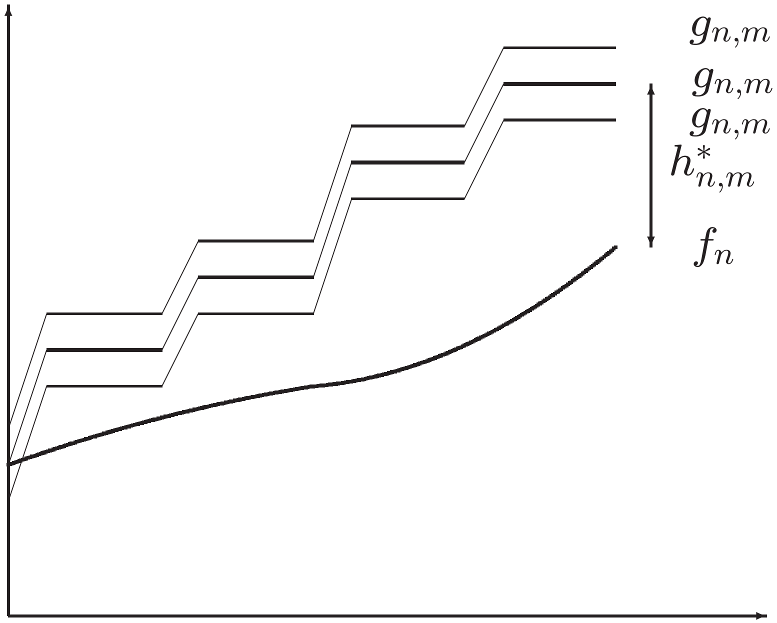


Assume $m \in \mathbb{N}$ is given.

We choose a countable dense subset, $\{f_n\}$, in $\mathcal{M}^1$, such that each f_n is continuously differentiable and $f_n' > 0$ on $[0,1]$.

The auxiliary functions $g_n^*(x)$ and $g_{n,m}^{**}(x)$ are chosen.

Finally we put $g_{n,m}(x) = g_n^*(x) + g_{n,m}^{**}(x)$.



We have $|f_n(x) - g^*_n(x)| < \dfrac{1}{2n}$
and $0 \le g^{**}_{n,m}(x) \le g^{**}_{n,m}(1) = h^*_{n,m}$.
We also have $|f_n(x) - g_{n,m}(x)| \le |f_n(x) - g^*_n(x)| + h^*_{n,m} \le \dfrac{3}{4n}$
Using $\varrho_{n,m} = \dfrac{(h_{n,m})^m}{8} < \dfrac{h_{n,m}}{8}$
we set $G_{n,m} = B(g_{n,m}, \varrho_{n,m})$ and $H_m = \cup_{n=m}^{\infty} G_{n,m}$.
It is clear that H_m is open in $\mathcal{M}^1$ and using the density of the functions $\{f_n\}$ and it is easy to see that H_m is dense.

To prove the main result we use the residual set $\mathcal{F}^* = \cap_{m=1}^{\infty} H_m$.

The higher dimensional results (joint work with S. Seuret):
An upper estimate of the singularity spectrum which is valid for arbitrary functions in $\mathcal{M}^d$.

T.: *For all* $f \in \mathcal{M}^d$ *and* $h \geq 0$, *we have*

$$\dim_H E_f^{h,\leq} \leq \min(d-1+h, d).$$

In particular, $d_f(h) = \dim_H(E_f^h) \leq \min(d-1+h, d)$.

The next theorem shows that for $h \in [0,1]$ the generic functions can be as bad as possible from the multifractal standpoint:

T.: *There exists a dense* G_δ *set* $\mathcal{R} \subset \mathcal{M}^d$ *such that for all* $f \in \mathcal{R}$ *we have* $d_f(h) = d - 1 + h$ *for all* $h \in [0,1]$. *For these functions, for every* $h > 1$ *the set* E_f^h *is empty.*

Level sets of MISV functions.

We define for every $a \in \mathbb{R}$ the level set $L_f(a)$ by

$$L_f(a) = \{x \in [0,1]^d : f(x) = a\}.$$

It is easy to see that for any continuous function the Hausdorff dimension of the level sets $L_f(a)$ in the interior of the range of an $f \in \mathcal{M}^d$ is at least $d-1$, for every a.

T.: *There exists a dense G_δ subset $\mathcal{L}$ in $\mathcal{M}^d$ such that for all $f \in \mathcal{L}$ the following holds.*
There exist a set $X_f \subset [0,1]^d$ and a set $A_f \subset (f(0,...,0), f(1,...,1)) = (m_f, M_f)$ satisfying:

- $\dim_H X_f = d-1$, $\dim_H A_f = 0$,
- *for every $a \in (m_f, M_f)$, there is at most one point of $L_f(a)$ which does not belong to X_f (in other words, $L_f(a) \cap ([0,1]^d \setminus X_f)$ contains at most one point).*
- *for every $a \in (m_f, M_f) \setminus A_f$, $L_f(a) \subset X_f$.*

Recall:

T.: *There exists a dense G_δ subset $\mathcal{L}$ in $\mathcal{M}^d$ such that for all $f \in \mathcal{L}$ the following holds.*
There exist a set $X_f \subset [0,1]^d$ and a set $A_f \subset (f(0,...,0), f(1,...,1)) = (m_f, M_f)$ satisfying:

- $\dim_H X_f = d-1$, $\dim_H A_f = 0$,
- *for every $a \in (m_f, M_f)$, there is at most one point of $L_f(a)$ which does not belong to X_f (in other words, $L_f(a) \cap ([0,1]^d \backslash X_f)$ contains at most one point).*
- *for every $a \in (m_f, M_f) \backslash A_f$, $L_f(a) \subset X_f$.*

In other words, X_f contains Lebesgue-almost every level sets $L_f(a)$, and for those level sets $L_f(a)$ which are not entirely contained in X_f
(this occurs for a set of values of a of Hausdorff dimension 0),
exactly one point of $L_f(a)$ does not belong to A_f.
This entails that our function f is "increasing" only on the small $d-1$ dimensional set X_f which has the minimum possible dimension to contain at least one level set.
Most points in the domain of f belong to $[0,1]^d \backslash X_f$, which can intersect just "very few" level sets and in no more than one point.
In particular, for all $x, x' \in [0,1]^d \setminus X_f$ (this set has full Lebesgue measure in $[0,1]$), $f(x) \neq f(x')$.

Convex hull of typical continuous functions

A. Bruckner and J. Haussermann

Let f be a bounded function on $[0, 1]$ and let H be the convex hull of the graph of f.
The boundary of H can be decomposed into the graphs of two functions ϕ_1 and ϕ_2, of which ϕ_1 is convex and ϕ_2 is concave.
These functions are differentiable except, perhaps, on some denumerable sets.
Even if f is well-behaved, say a Lipschitz function, the infinite exceptional sets of nondifferentiability may exist.
Surprisingly, for typical continuous f the functions ϕ_1 and ϕ_2 have finite derivatives everywhere on $(0, 1)$ and infinite derivatives at 0 and at 1.
T.:The set of functions in $C[0, 1]$ for which ϕ_1 and ϕ_1 have the properties listed below is residual in $C[0, 1]$.

- ϕ_1' and ϕ_2' exist and are continuous on $(0, 1)$;
- $\phi_1'(0) = -\infty$, $\phi_1'(1) = +\infty$, $\phi_2'(0) = +\infty$, and $\phi_2'(1) = -\infty$;
- ϕ_1' and ϕ_2' are unbounded Cantor-like functions.

Z.B. and S. Seuret (work in progress):
T.:The Hölder spectrum of of ϕ_j, $(j = 1, 2)$: $E^h_{\phi_j} = \emptyset$, that is $d_{\phi_j}(h) = -\infty$, for $0 < h < +\infty$, $d_{\phi_j}(+\infty) = \dim_H E^{+\infty}_{\phi_j} = 1$, $d_{\phi_j}(0) = 0$.

Generic convex continuous functions

Denote by $\mathcal{C}^d$ the (closed) subspace of convex continuous functions in $C[0,1]^d$.

Z.B. and S. Seuret (work in progress):
There is a residual subset $\mathcal{B} \subseteq \mathcal{C}^1$ with the following properties:
• $\mathcal{B}$ consists of continuously differentiable functions and
$f''(x) = 0$, for a.e. $x \in [0,1]$;
• the Hölder singularity spectrum of an $f \in \mathcal{B}$ is given by
$d_f(h) = -\infty$, for $h \in [0,1) \cup (2,+\infty]$, and $d_f(h) = h - 1$, for $h \in [1,2]$.

The higher dimensional version:
Z.B. and S. Seuret (work in progress):
There is a residual subset $\mathcal{B} \subseteq \mathcal{C}^p$ with the following properties:
• $\mathcal{B}$ consists of continuously differentiable functions and
$f''(x) = 0$, for a.e. $x \in [0,1]^p$;
• the Hölder singularity spectrum of an $f \in \mathcal{B}$ is given by
$d_f(h) = -\infty$, for $h \in [0,1) \cup (2,+\infty]$, and $d_f(h) = h + d - 2$, for $h \in [1,2]$.

Level sets of generic continuous functions, topological approach

If X is a compact metric space, then $C(X,I)$ denotes the set of continuous functions from X into the unit interval I endowed with sup norm.
We recall that $C(X,I)$ is a Polish space.
A *map* is simply a continuous function.

The Bruckner-Garg Theorem:

T.:*Suppose $X=[0,1]$. A generic $f\in C(X,I)$ has the property that there is a countable dense set $D\subseteq(\min f,\max f)$ such that*
(i) $f^{-1}(y)$ is a singleton set if $y\in\{\min f,\max f\}$,
(ii) $f^{-1}(y)$ is homeomorphic to a Cantor set when $y\in(\min f,\max f)\setminus D$,
(iii) $f^{-1}(y)$ is homeomorphic to the union of a Cantor set and an isolated point when $y\in D$.

The Bruckner-Grag theorem was generalized by Z.B. and U.B. Darji to generic/typical continuous functions in $C(S^2,I)$.
We use S^2 to denote the 2-sphere in $\mathbb{R}^3$.
We point out $C(S^2,I)$ is homeomorphic to the set of all continuous functions $f:\mathbb{R}^2\to I$ which have a limit at infinity.

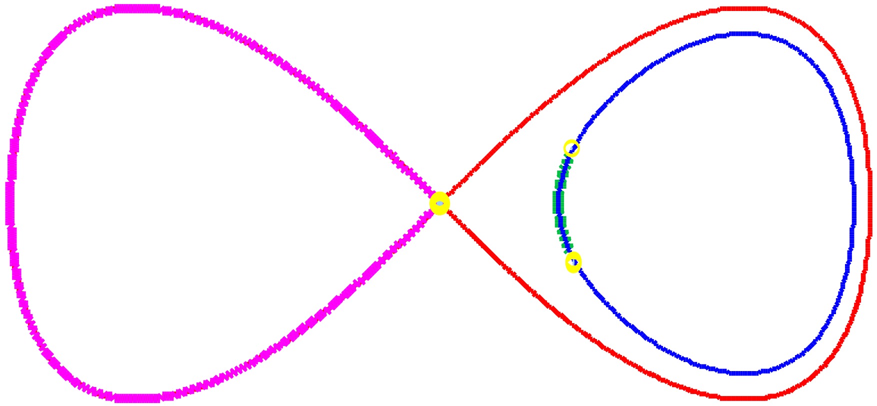
A *continuum* is a compact connected metric space. A continuum is *degenerate* if it has only one point. Otherwise, we say that it is nondegenerate.

A continuum is *decomposable* if it is the union of two proper subcontinua. Otherwise we call it *indecomposable*. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

(On the figure: The buckethandle, or B-J-K continuum (for Brouwer, Janiszewski and Knaster) which is an indecomposable plane continuum. source: Wikipedia.)

By fiber we mean non-empty level sets.

T.:(Krasinkiewicz-Levin) Let X be a compact metric space. Then, a generic $f \in C(X, I)$ has the property that each of its fibers is a Bing compactum, a compactum with all components hereditarily indecomposable.



A circle is figure eight-like

A map f from a metric space X onto a metric space Y is an ϵ-map means that $\epsilon > 0$ and $diam(f^{-1}(y)) < \epsilon$ for every $y \in Y$.
We say that continuum X is *P-like* if for every $\epsilon > 0$, there is an ϵ-map from X onto P.

A continuum which is arc-like (circle-like) is often called *chainable* (*circularly chainable*). Figure-eight is a continuum homeomorphic to the union of two circles which intersect in exactly one point (and it is not circle-like).

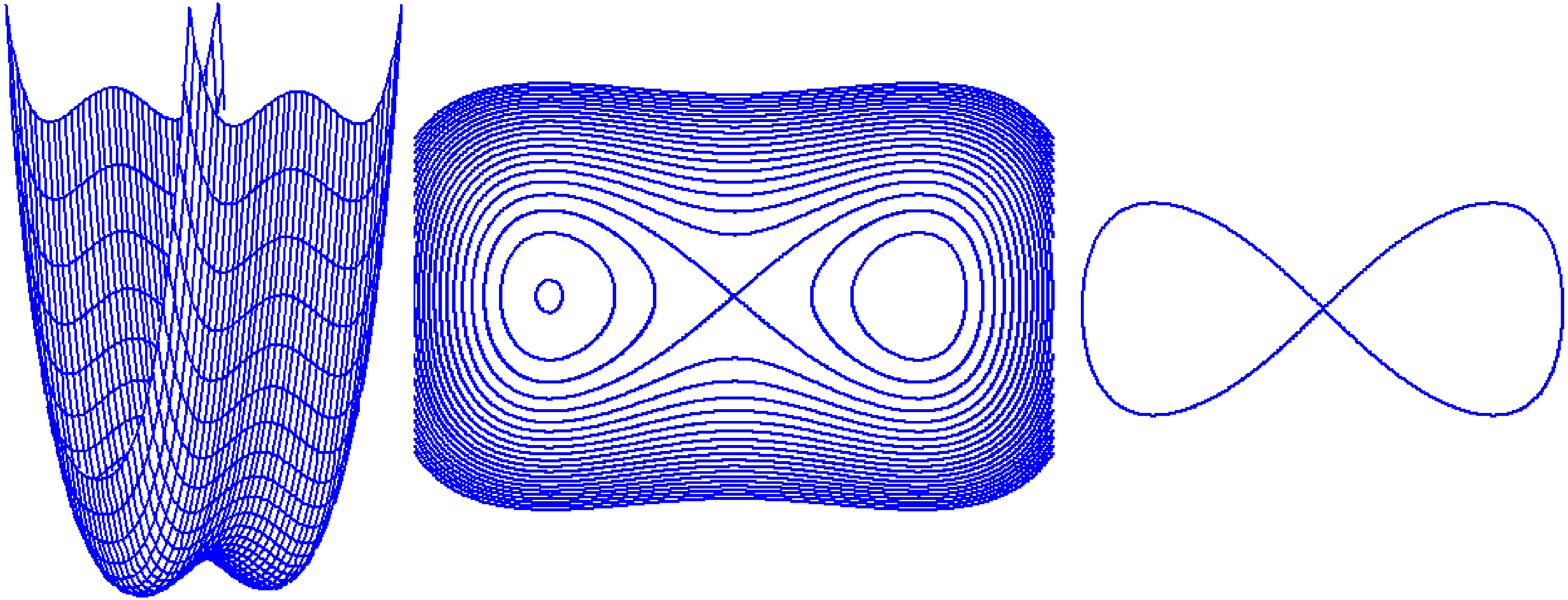
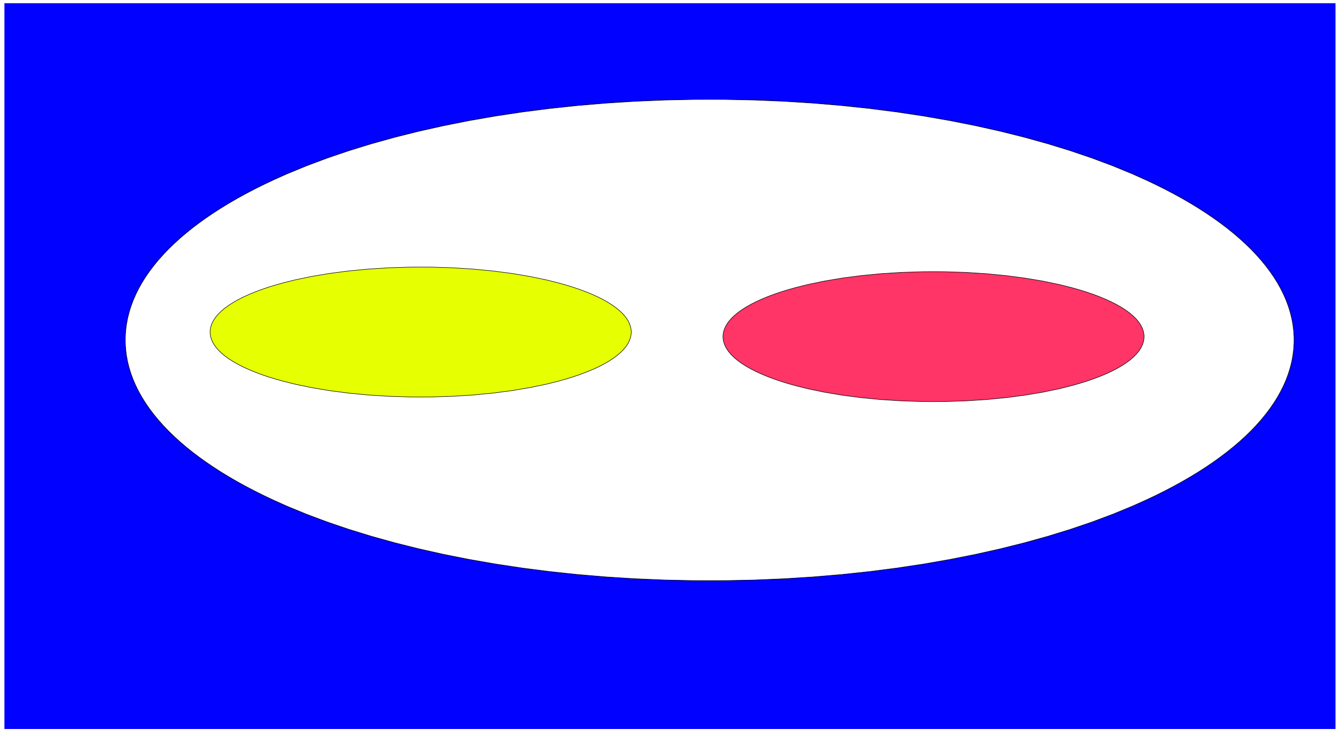


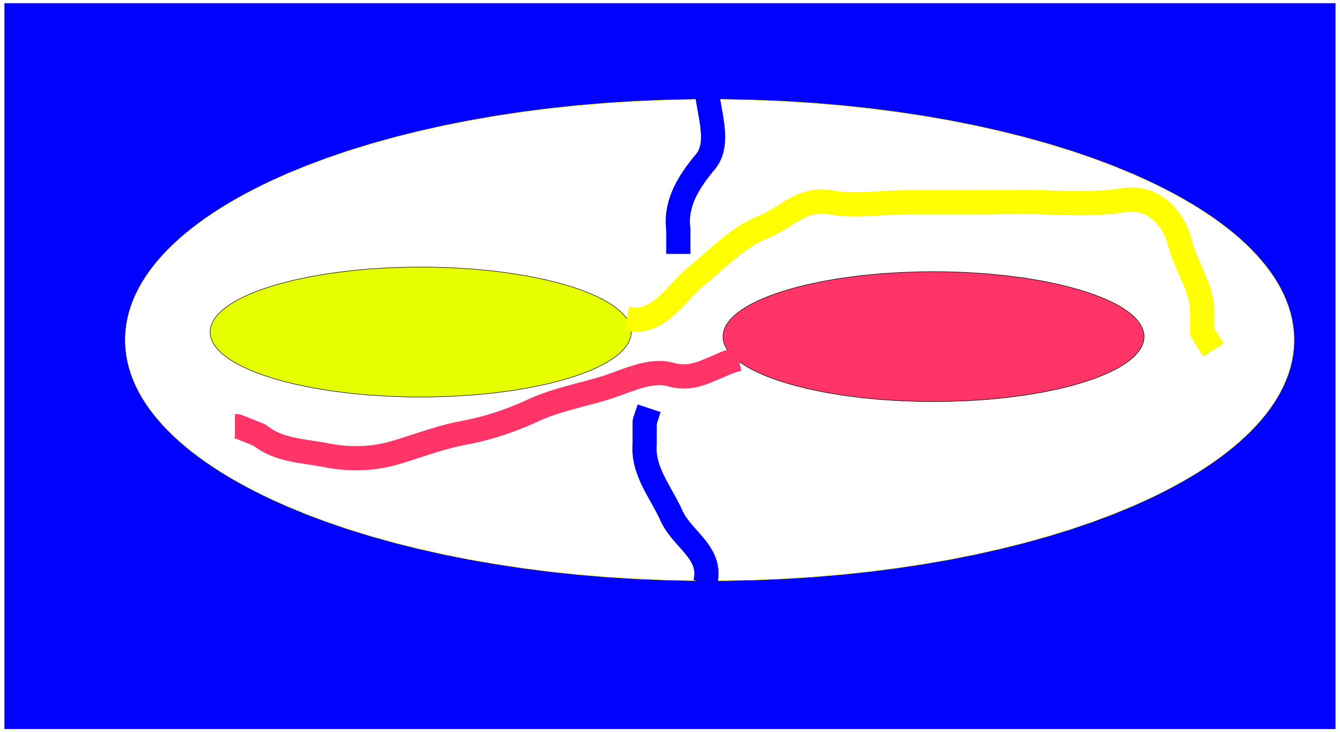
Figure eight levels show up at saddle points for smooth maps.



A *pseudoarc* is a hereditarily indecomposable continuum which is arc-like. We recall that up to homeomorphism the pseudoarc is unique.

There are uncountably many nonhomeomorphic pseudocircles, that is hereditarily indecomposable continua which are circle-like. However, in the plane or S^2, up to homeomorphism, there is only one pseudocircle.

We call a continuum $X \subseteq S^2$ *a Lakes of Wada continuum* if it is hereditarily indecomposable, $S^2 \setminus X$ has exactly three components and X is the boundary of each of these components.



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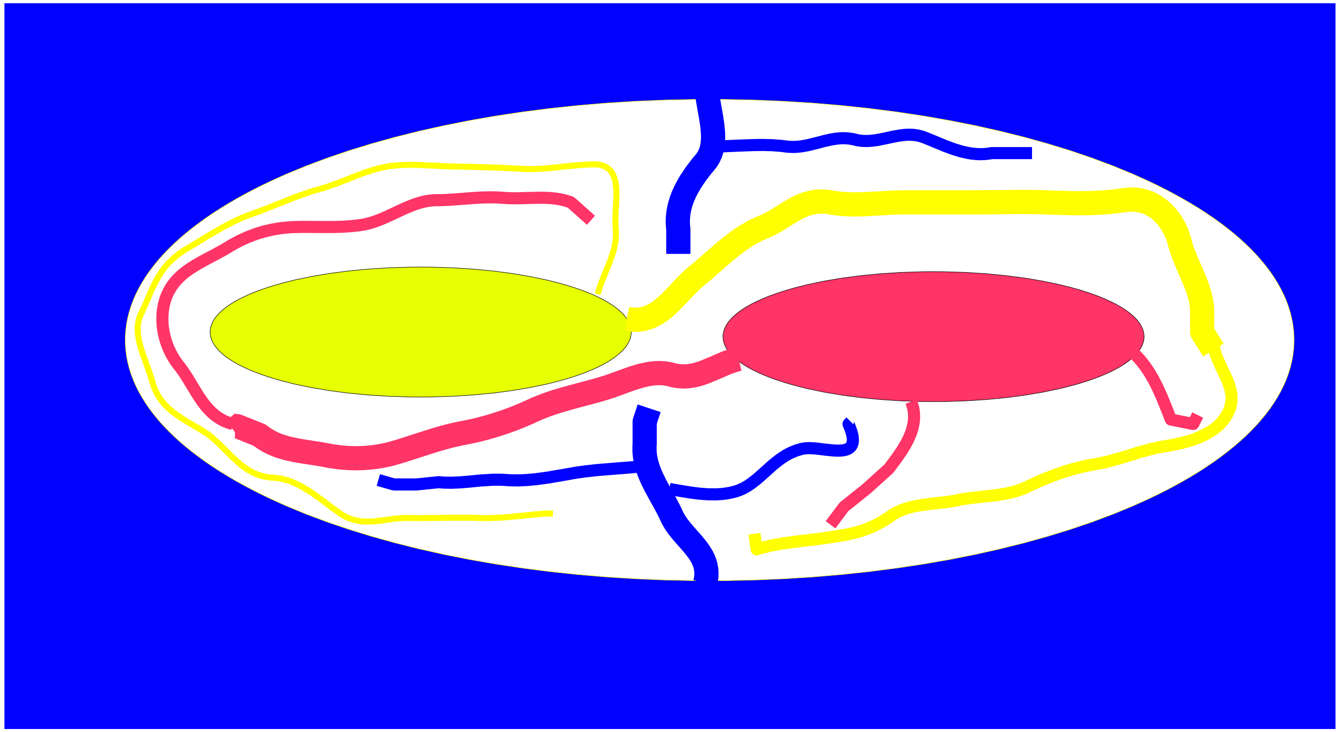

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The next theorems are from a joint paper with U. B. Darji

T.:*A generic* $f \in C(S^2, I)$ *has the property that each component of each fiber of* f *is either a point, or a hereditarily indecomposable continuum which is figure-eight-like.*

T.:*A generic* $f \in \mathcal{C}(S^2, I)$ *has the property that if*

$$y \in (\min(f(S^2)), \max(f(S^2))),$$

then $f^{-1}(y)$ *contains a component which is a pseudoarc.*

T.:*A generic* $f \in \mathcal{C}(S^2, I)$ *has the property that there is a countable dense set* $D \subseteq f(S^2)$ *such that for each* $y \in f(S^2)$ *every component of* $f^{-1}(y)$ *separates* S^2 *into two pieces or less except when* $y \in D$. *In the latter case, the same applies to each component with one exception which separates* S^2 *into exactly three pieces.*

The above theorems imply that for $y \in D$ one component of the level set $f^{-1}(y)$ is a Lakes of Wada continuum. These Lakes of Wada continua correspond to the "saddle points" of generic continuous functions of two variables.

Level sets of generic continuous functions, measure theoretical approach

B. Kirchheim:

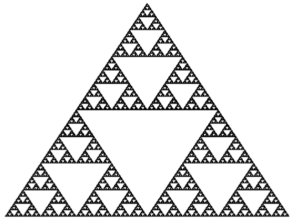
T.:*Let* $1 \leq n < m$ *be given. For any* $k \geq 1$ *we write* $d_k = m - k(m-n)$. *Then for a typical (generic) continuous mapping* f *of the unit ball* $B(0,1)$ *of* $\mathbb{R}^n$ *into* $\mathbb{R}^m$, *and for any* $k \geq 2$, *the set* $M_k = \{x \in B(0,1): \ \operatorname{card} f^{-1}(f(x)) \geq k\}$ *as well as the set* $f(M_k)$ *are* F_σ *sets of Hausdorff dimension* d_k; *moreover, for any nonempty (open) set* $U \subset B(0,1)$, *both sets* $U \cap M_k$ *and* $f(U \cap M_k)$ *are of non-*σ*-finite* d_k*-dimensional measure. For* $k = 1$ *the statement concerning* $f(U \cap M_1)$ *remains true.*

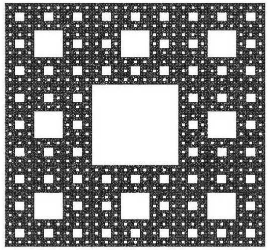
T.:*Let* $n \geq m \geq 1$. *Then for a typical (generic) continuous* $f : [0,1]^n \to R^m$:
(i) $\operatorname{int}(\operatorname{im} f) \neq \emptyset$, $\partial(\operatorname{im} f)$ *is of Hausdorff dimension* $m-1$;
(ii) for any $y \in \mathbb{R}^m$ *the level set* $f^{-1}(y)$ *is of Hausdorff dimension at most* $n-m$, *and it is of non-*σ*-finite* $\mathcal{H}^{n-m}$ *measure whenever* $y \in \operatorname{int}(\operatorname{im} f)$.

Especially when $n = 2$ and $m = 1$, that is we have a map $f : [0,1]^2 \to \mathbb{R}$ then $f^{-1}(y)$ is of Hausdorff dimension at most 1, and it is of non-σ-finite $\mathcal{H}^1$ measure whenever $y \in \operatorname{int}(\operatorname{im} f)$.

Level sets of generic continuous functions on fractals

Schweitzer competition 2008, problem 8:





Suppose S is the Sierpinski triangle. For the generic continuous function on S determine the Hausdorff dimension of the level sets $f^{-1}(y)$.

More difficult version (not accepted by the committee):

Suppose S is the Sierpinski carpet. For the generic continuous function on S determine the Hausdorff dimension of the level sets $f^{-1}(y)$.

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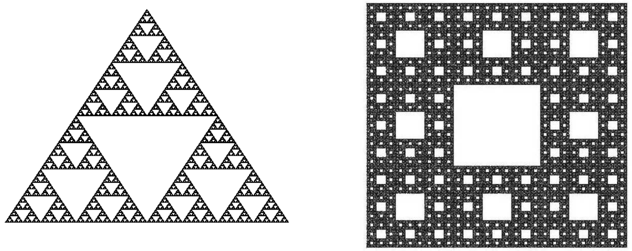

Suppose S is the Sierpinski triangle. For the generic continuous function on S determine the Hausdorff dimension of the level sets $f^{-1}(y)$.

Answer: $\dim f^{-1}(y) = 0$ for all $x \in f(S)$.

More difficult version (not accepted by the committee):

Suppose S is the Sierpinski carpet. For the generic continuous function on S determine the Hausdorff dimension of the level sets $f^{-1}(y)$.

Answer: $\dim f^{-1}(y) = \log 2 / \log 3$ for all $x \in \mathrm{int} f(S)$, and $f^{-1}(y)$ consists of one point if $y \in \{\max f(S), \min f(S)\}$.

A graduate student R. Balka got interested in my Schweitzer problem the next results are from a joint work with him and M. Elekes :

T.:*If K is a compact metric space then there exists exactly one number $d(K) \geq 0$ such that*
(i) the Hausdorff dimension of each level set of the generic $f \in C(K)$ is at most $d(K)$;
(ii) For the generic $f \in C(K)$ for any $\epsilon > 0$ there exists a non-degenerate interval $I_{f,\epsilon}$ such that for any $y \in I_{f,\epsilon}$ we have $\dim(f^{-1}(y)) \geq d(K) - \epsilon$.

D.:Suppose K is a compact subset in a metric space. We say that K is weakly self similar if there exists $r_0 > 0$ such that for all $x \in K$ and $0 < r < r_0$ there exists $K_{x,r} \subseteq B(x, r)$ and $\phi_{x,r} : K \to K_{x,r}$ bilipschitz onto map.

T.:*Suppose that the compact metric space K satisfies the above weak self similarity property. Then for the generic continuous function $f \in C(K)$ for any interior point, y of $f(K)$ we have $\dim(f^{-1}(y)) = d(K)$.*

Homeomorphic restrictions of continuous functions

The usual metrics ρ_0, and ρ_1 on $C[0,1]$, and on $C^1[0,1]$, respectively, are given by
$\rho_0(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$ for $f,g \in C[0,1]$, and
$\rho_1(f,g) = \rho_0(f,g) + \rho_0(f',g')$ for $f,g \in C^1[0,1]$.
$B_0(f,r)$ and $B_1(f,r)$ are the open balls in the metrics ρ_0 and ρ_1 respectively.
Suppose $F \subset [0,1]$ nowhere dense, perfect, nonempty.

Prop.: *The typical $f \in C[0,1]$ is a homeomorphism on F.*
More is true.
S. Agronsky, A.M. Bruckner and M. Laczkovich: Dynamics of typical continuous functions:
T.: *Let $F \subset [0,1]$ be of first category. Then $\exists\ \mathcal{H} \subset C[0,1]$ residual, such that $\forall f \in \mathcal{H}$ the sets $f^n(F)$, $n = 0,1,...$ are pairwise disjoint and f is one-to-one on each set, $f^n(F)$, $n = 0,1,....$*

The next results are from a joint paper with A. Máthé:
Let F be a closed set in $[0,1]$. Consider the Cartesian product $F \times F$, and its projections in various directions.
Let us denote by $\pi_{\beta/\alpha}$ the projection onto the line with tangent vector (α, β) of unit length, that is, $\alpha^2 + \beta^2 = 1$ and
$\pi_{\beta/\alpha}(x, y) = \alpha x + \beta y$.
Note that β/α is the slope of the line with tangent vector (α, β).

We say that **property $\mathcal{P}$ holds** for the closed set $F \subset [0,1]$
if there exists a dense subset H of $\mathbb{R}$ for which $\pi_h(F \times F) \subset \mathbb{R}$ is nowhere dense for every $h \in H$.
That is, the image of $F \times F$ is nowhere dense under projections in some "dense set of directions".

T.: *Let $F \subset [0,1]$ be a closed set. If property $\mathcal{P}$ holds for F then the typical $C^1[0,1]$ function is one-to-one on F.*

T.: *If* $F \subset [0,1]$ *is closed and* $\underline{dim}_B F < 1/2$ *then a typical* $C^1[0,1]$ *function is one-to-one on* F*.*

T.: *Let* F *be a closed subset of* $[0,1]$*. If the Hausdorff dimension of* $F \times F$ *is less than one then a typical* $C^1[0,1]$ *function is one-to-one on* F*.*

T.: *Let* $F \subset [0,1]$ *be a self-similar set with OSC of dimension* $\leq 1/2$*. Then a typical* $C^1[0,1]$ *function is injective on* F*.*

D.: Suppose $0 < \alpha < 1$ and $t = (1-\alpha)/2$. The middle-α Cantor set, denoted by C_α, is the self-similar set generated by the similarities
$\phi_1 : x \mapsto \frac{1-\alpha}{2}x = tx$ and
$\phi_2 : x \mapsto 1 + (x-1)\frac{1-\alpha}{2} = (1-t) + tx$.
When $\alpha = 1/3$ we obtain the usual triadic Cantor set.
Let Φ be the operator on compact subsets of $\mathbb{R}$ for which
$\Phi(F) = \phi_1(F) \cup \phi_2(F)$.
Put $F_n = \Phi^n([0,1])$, $(n = 0, 1, \ldots)$, which is a union of 2^n intervals of length t^n.
Then $C_\alpha = \bigcap_{n=0}^{\infty} F_n$.

T.: *A typical $C^1[0,1]$ function is injective on C_α if and only if* $\dim(C_\alpha) \leq 1/2$ *(that is, $1/2 \leq \alpha < 1$).*

T.: *There exists a closed set $F \subset [0,1]$ of Hausdorff dimension one such that a typical $C^1[0,1]$ function is one-to-one on F.*

T.: *There exists a closed $F \subset [0,1]$ such that $dim_B F = 1/2$ and the set of those $f \in C^1[0,1]$ for which $f|_F$ is one-to-one is not dense in $C^1[0,1]$.*